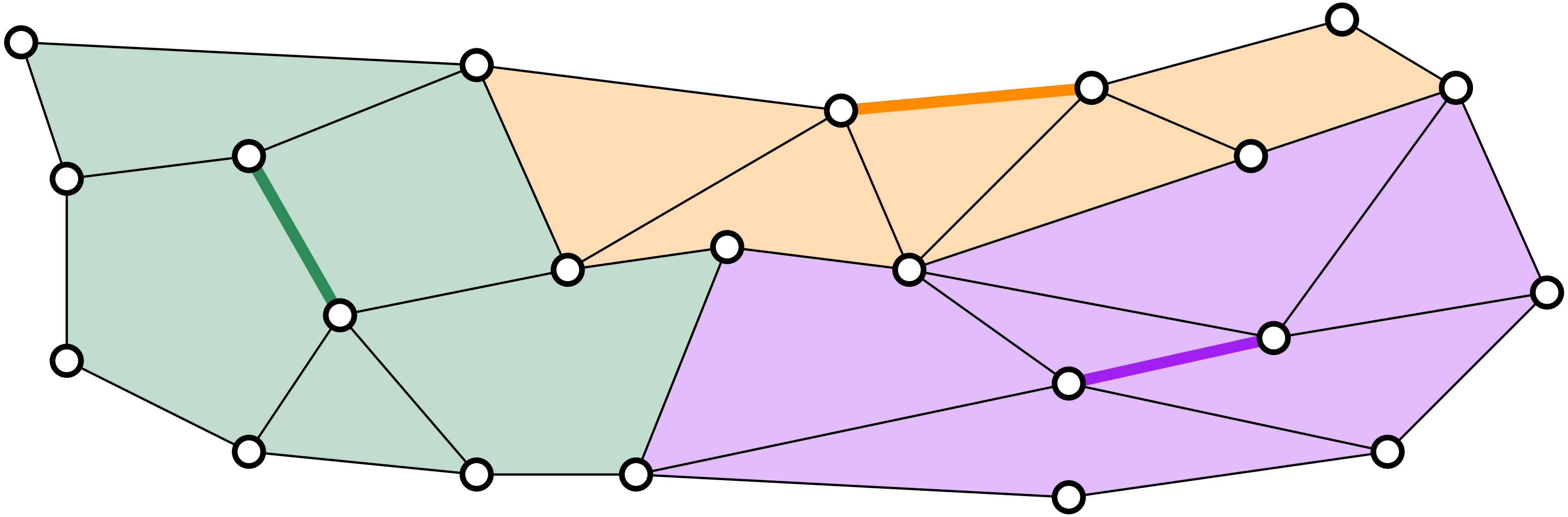
Karlsruhe Institute of Technology

Guarding Quadrangulations and Stacked Triangulations with Edges

June 24, 2020
Paul Jungeblut, Torsten Ueckerdt

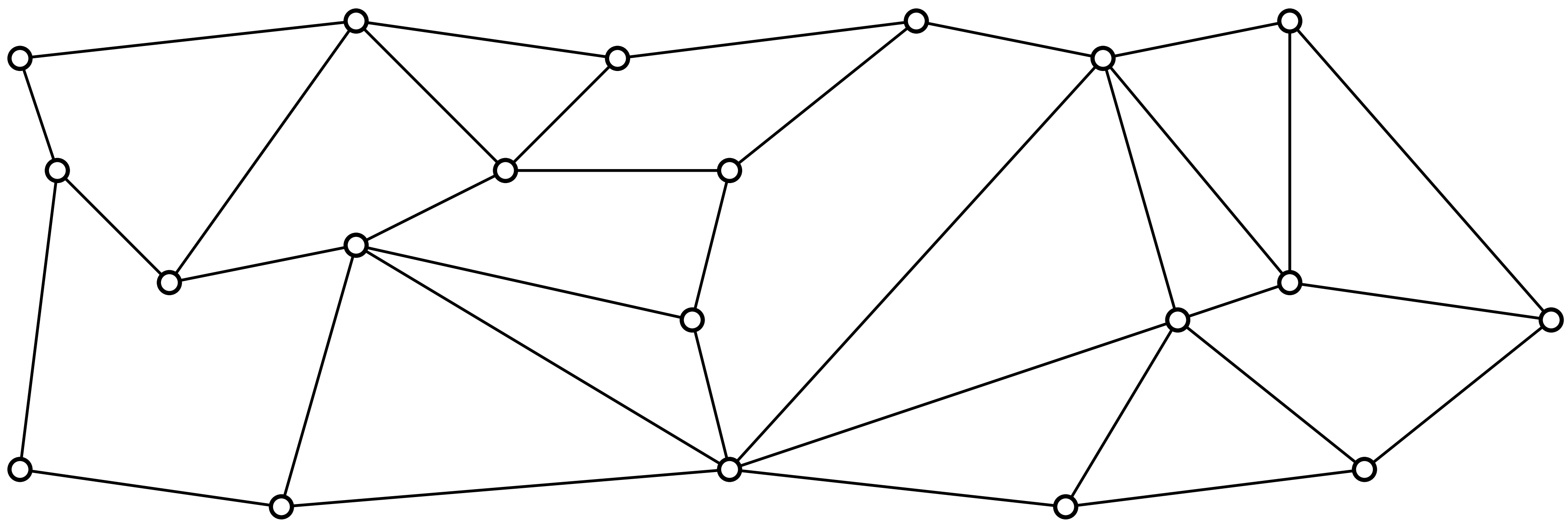
INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS GROUP



Edge Guarding

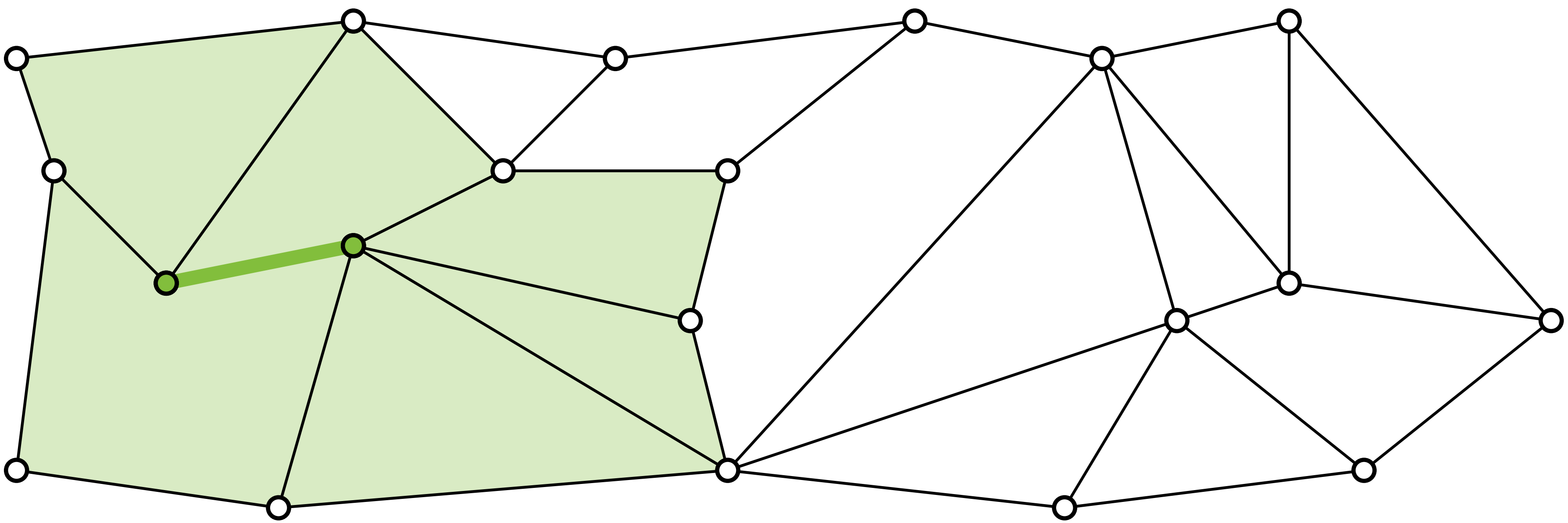
Karlsruhe Institute of Technology

- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



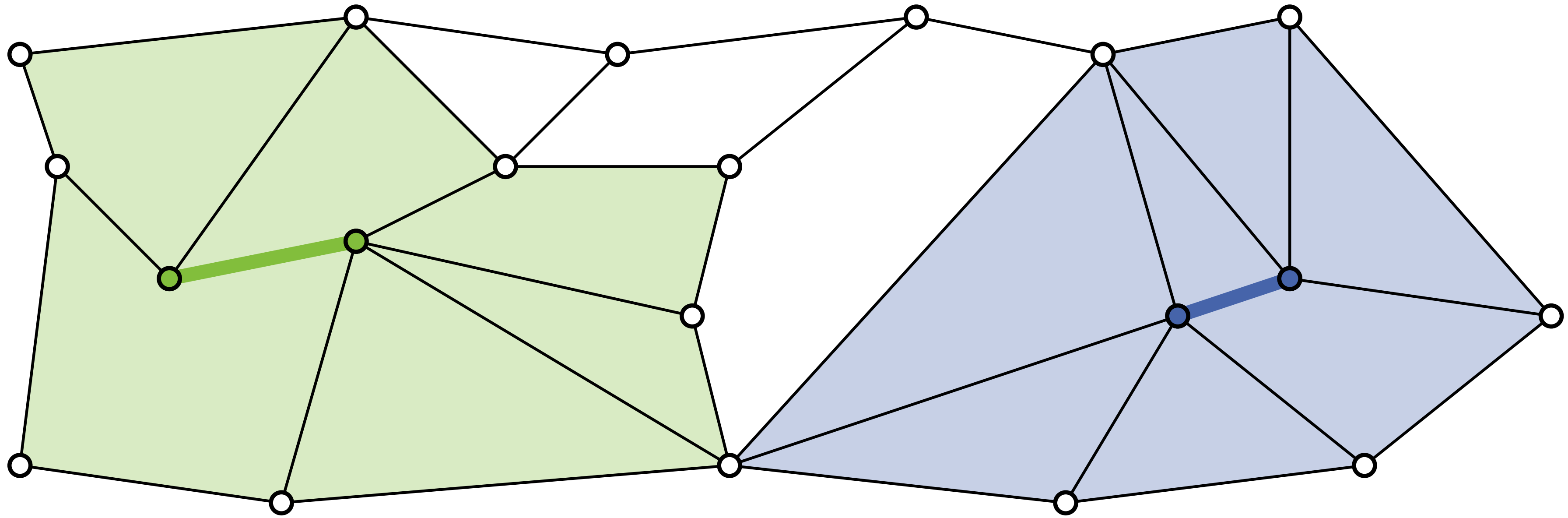
Edge Guarding

- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



Edge Guarding

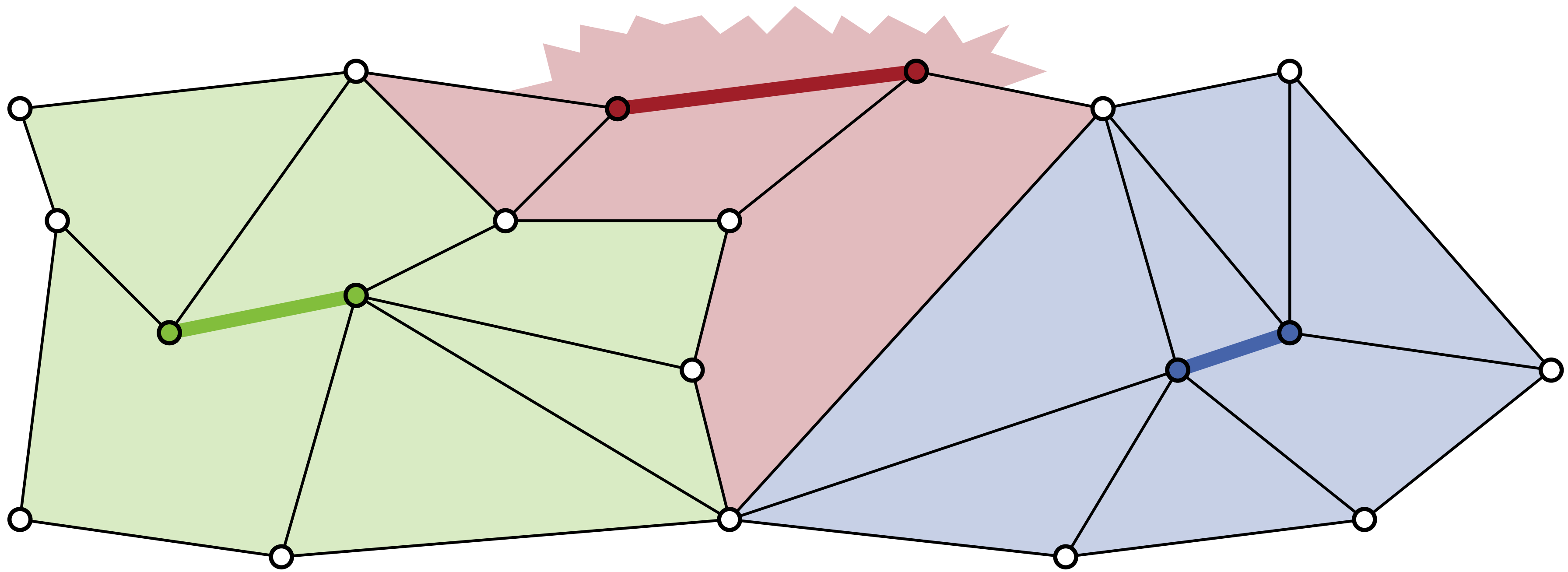
- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



Institute of Theoretical Informatics
Algorithmics Group

Edge Guarding

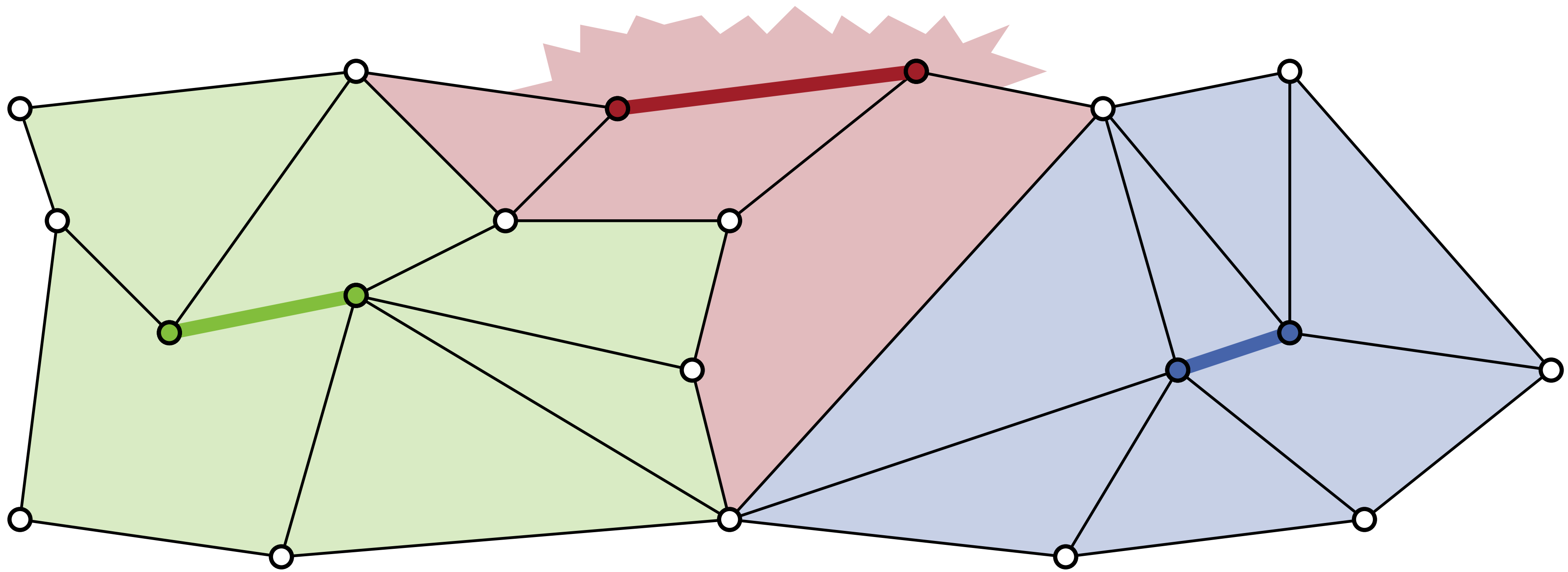
- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



Institute of Theoretical Informatics
Algorithmics Group

Edge Guarding

- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



Question

For all n-vertex graphs of a planar graph class $\mathcal{C}$:
How many guards are sometimes necessary and always sufficient?

Previous Results

Karlsruhe Institute of Technology

	Lower	Upper
Planar	$\left\lfloor \frac{n}{3} \right\rfloor$[1]	$\min\left\{ \left\lfloor \frac{3n}{8} \right\rfloor, \left\lfloor \frac{n}{3} + \frac{\alpha}{9} \right\rfloor \right\}$[2]
Triangulation	$\left\lfloor \frac{4n-8}{13} \right\rfloor$[1]	$\left\lfloor \frac{n}{3} \right\rfloor$[3]
Outerplanar	$\left\lfloor \frac{n}{3} \right\rfloor$[1]	$\left\lfloor \frac{n}{3} \right\rfloor$[4]
Max. Outerplanar	$\left\lfloor \frac{n}{4} \right\rfloor$[5]	$\left\lfloor \frac{n}{4} \right\rfloor$[5]

α: number of quadrilateral faces

[1] Bose, Shermer, Toussaint, Zhu 1997
[2] Biniaz, Bose, Ooms, Verdonschot 2019
[3] Everett, Rivera-Campo 1997
[4] Chvátal 1975
[5] O'Rourke 1983

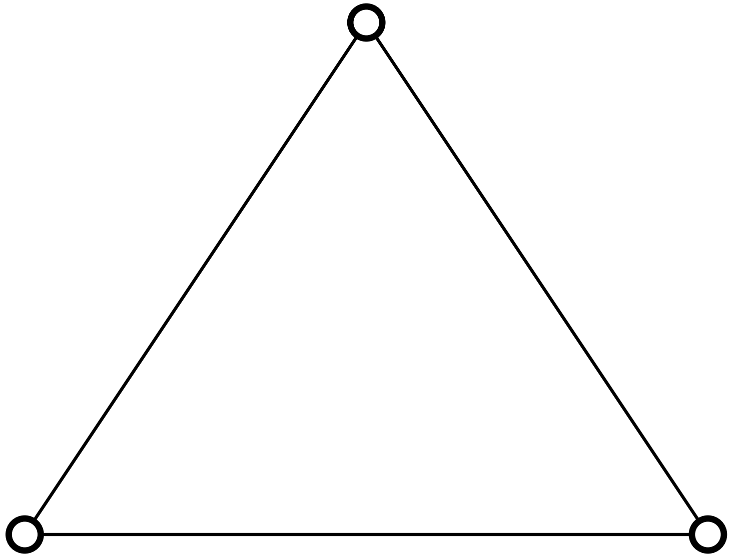
Our Results

	Lower	Upper
Stacked Triangulations	$\left\lfloor \frac{2n-4}{7} \right\rfloor$	$\left\lfloor \frac{2n}{7} \right\rfloor$
Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{3} \right\rfloor$
2-Degenerate Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$

Our Results

	Lower	Upper	
Stacked Triangulations	$\left\lfloor \frac{2n-4}{7} \right\rfloor$	$\left\lfloor \frac{2n}{7} \right\rfloor$	**Today!**
Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{3} \right\rfloor$	
2-Degenerate Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$	

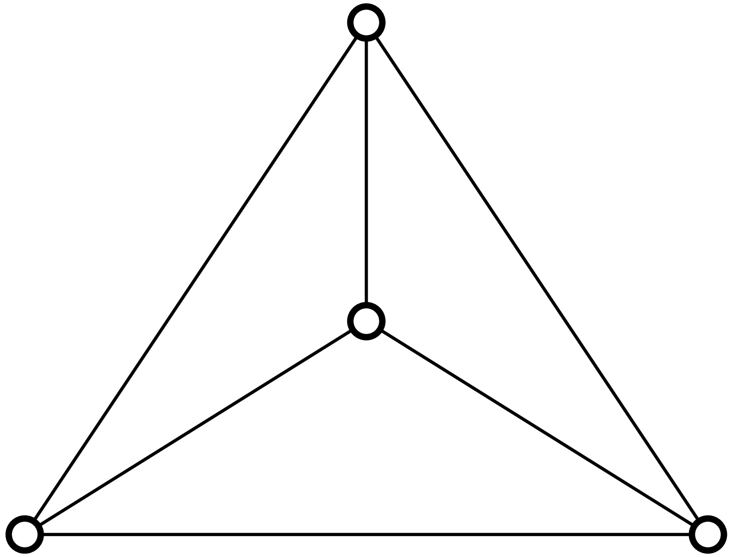
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation: Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

Institute of Theoretical Informatics
Algorithmics Group

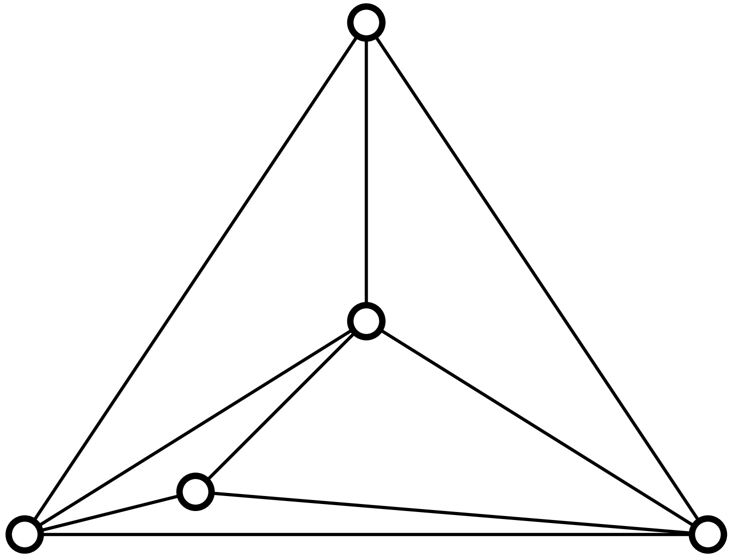
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation:
  Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

Institute of Theoretical Informatics
Algorithmics Group

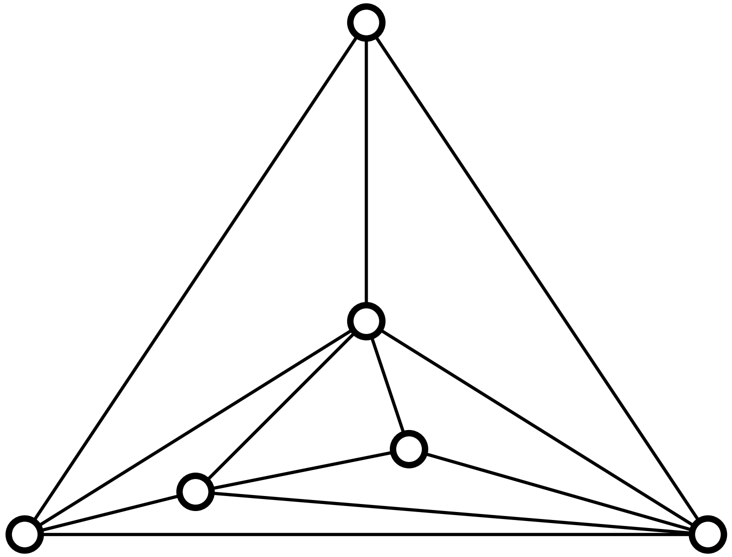
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let *f* be an inner face of a stacked triangulation:
  Adding a new vertex into *f* and subdividing it into three new faces gives a stacked triangulation.

Institute of Theoretical Informatics
Algorithmics Group

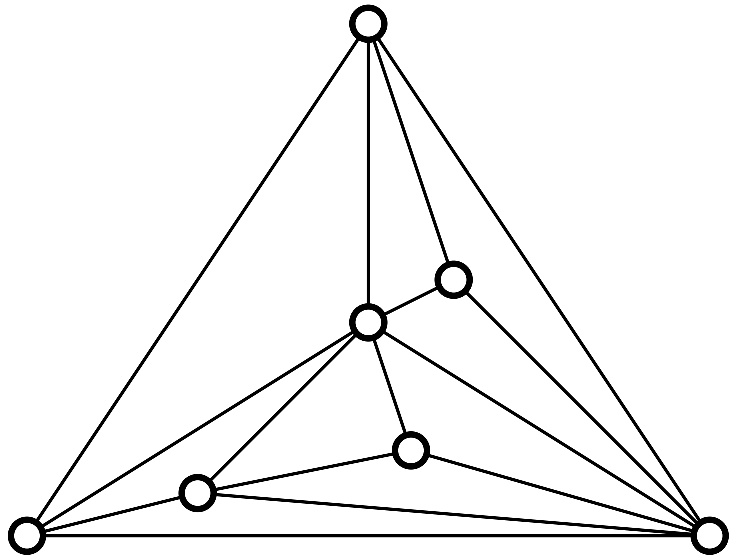
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation:
  Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

Institute of Theoretical Informatics
Algorithmics Group

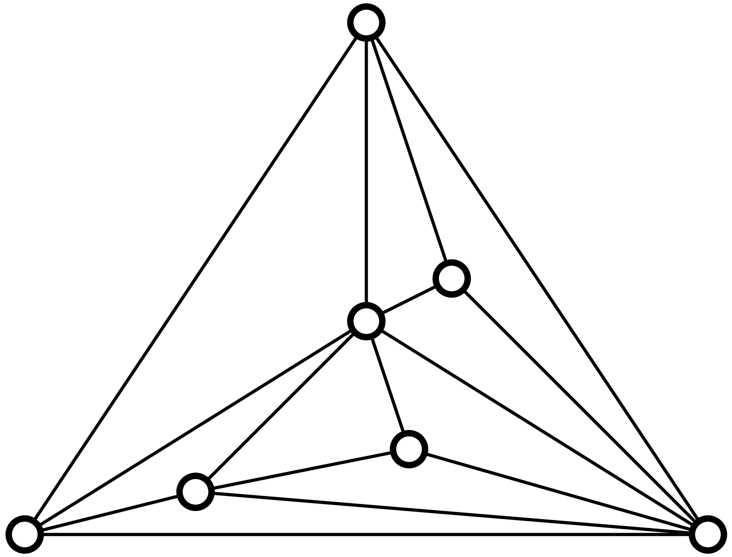
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation:
  Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

Institute of Theoretical Informatics
Algorithmics Group

Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation:
  Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

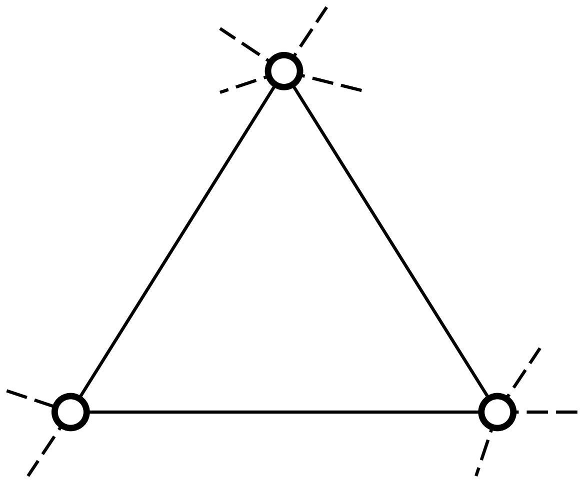
Theorem [J. 2019]

For n-vertex stacked triangulations $\left\lfloor \frac{2n}{7} \right\rfloor$ edge guards are sometimes necessary and always sufficient.

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

About $\frac{2n}{7}$ guards are sometimes necessary.



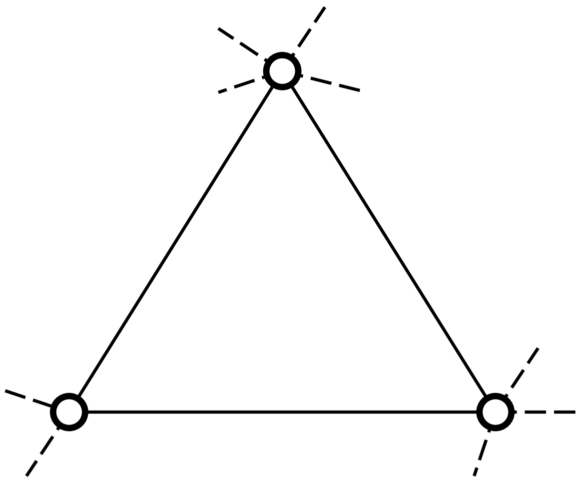
- S: v-vertex stacked triangulation ("skeleton")

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

Karlsruhe Institute of Technology

About $\frac{2n}{7}$ guards are sometimes necessary.

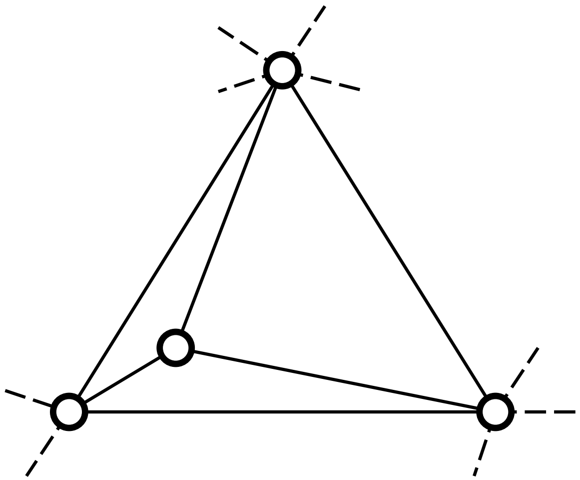


- S: v-vertex stacked triangulation ("skeleton")
- $S \leadsto G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

About $\frac{2n}{7}$ guards are sometimes necessary.

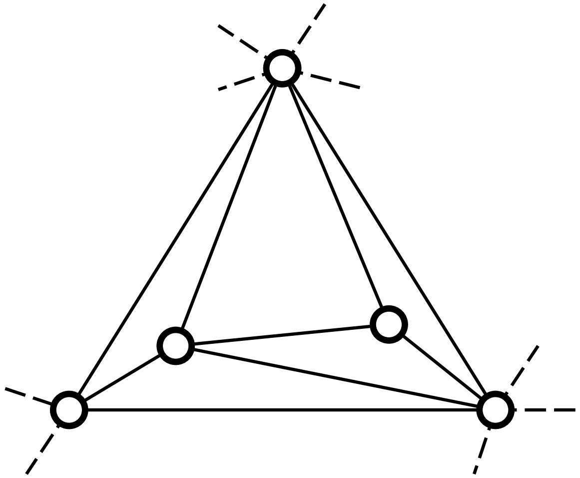


- S: v-vertex stacked triangulation ("skeleton")
- $S \rightsquigarrow G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

About $\frac{2n}{7}$ guards are sometimes necessary.

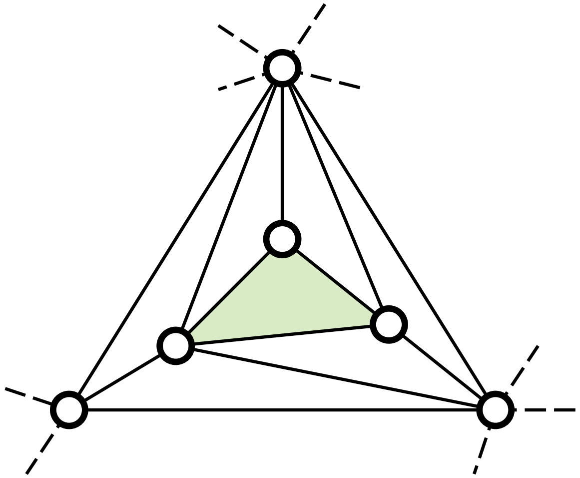


- S: v-vertex stacked triangulation ("skeleton")
- $S \rightsquigarrow G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

About $\frac{2n}{7}$ guards are sometimes necessary.



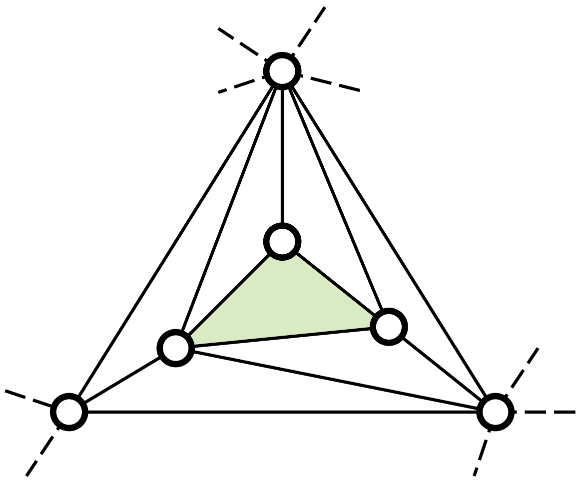
- S: v-vertex stacked triangulation ("skeleton")
- $S \rightsquigarrow G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

Karlsruhe Institute of Technology

About $\frac{2n}{7}$ guards are sometimes necessary.



- S: v-vertex stacked triangulation ("skeleton")
- $S \rightsquigarrow G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

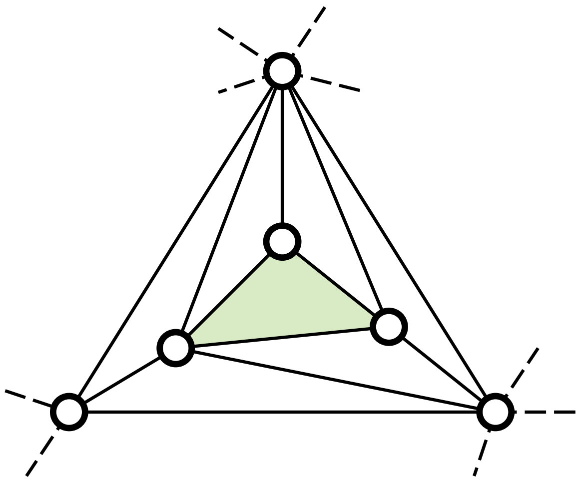
$$n = |V(G)| = v + 3 \cdot (2v - 4) = 7v - 12$$

Institute of Theoretical Informatics
Algorithmics Group

Lower Bound: Construction

Karlsruhe Institute of Technology

About $\frac{2n}{7}$ guards are sometimes necessary.



- S: v-vertex stacked triangulation ("skeleton")
- $S \rightsquigarrow G$: Add 3 vertices per face such that:
  - G is a stacked triangulation.
  - The new vertices form a triangular face.

$$n = |V(G)| = v + 3 \cdot (2v - 4) = 7v - 12$$

$$v = \frac{n+12}{7}$$

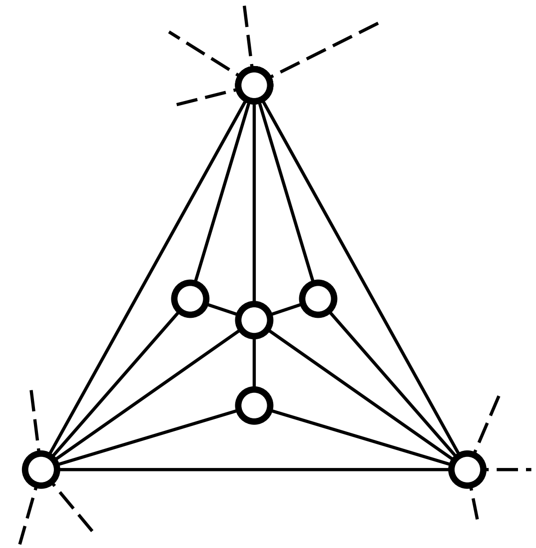
No two "green" triangles can share a guard.

$$|\text{guards}| \geq 2v - 4 = \frac{2n-4}{7}$$

Institute of Theoretical Informatics
Algorithmics Group

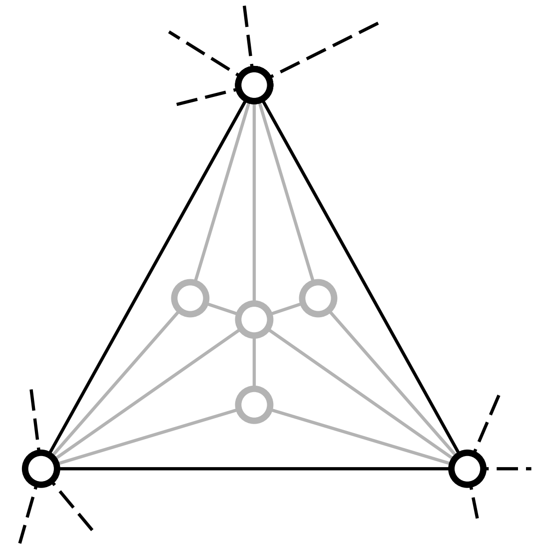
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



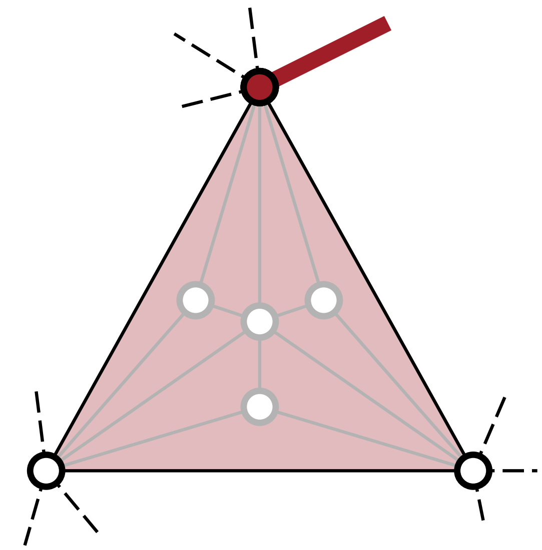
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



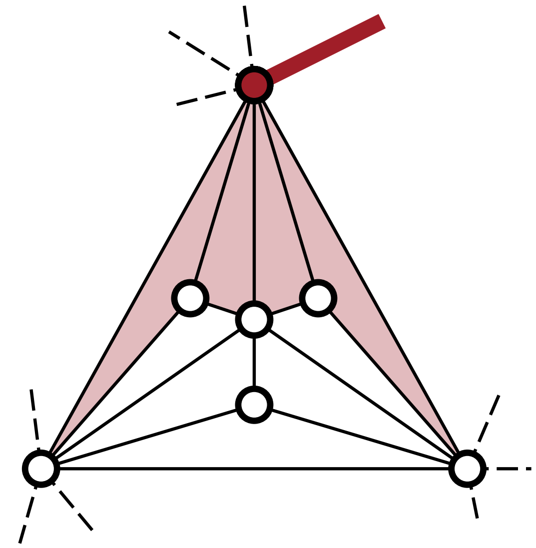
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



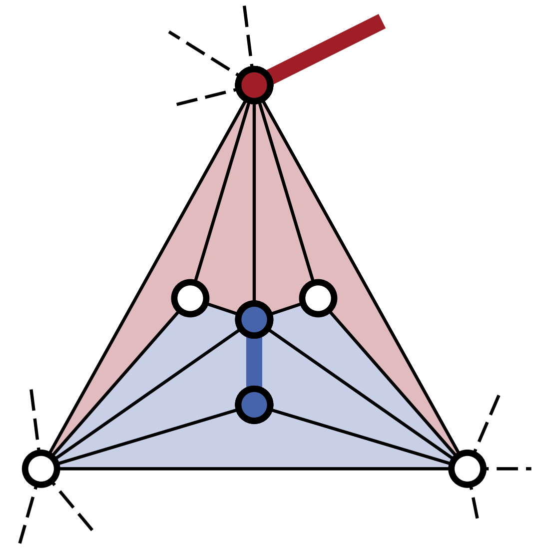
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



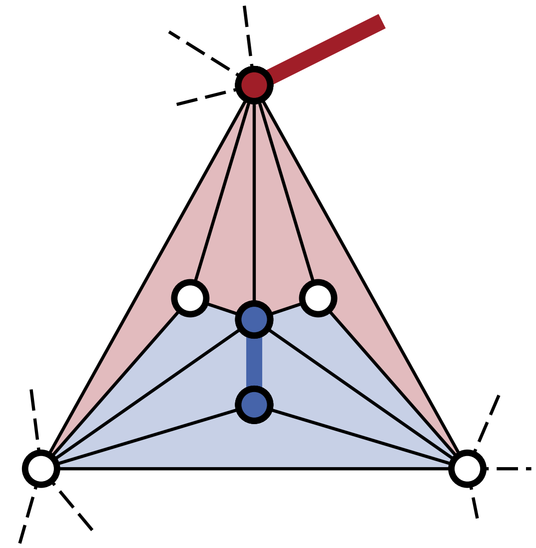
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



Upper Bound: Induction via Vertex Deletion

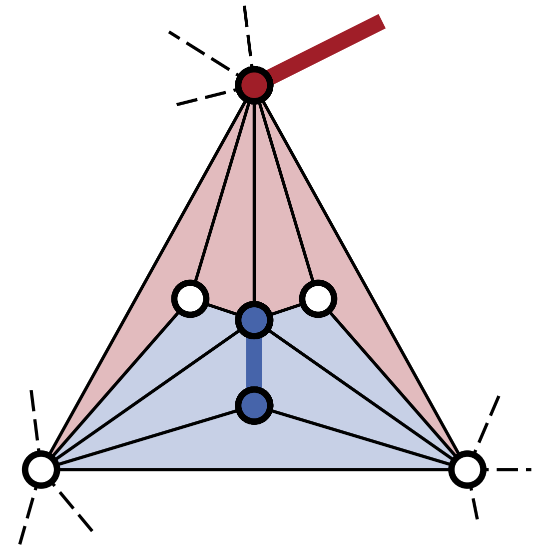
- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



- $\frac{\ell}{k} \leq \frac{2}{7}$ in all cases $\Rightarrow$ edge guard set of size $\left\lfloor \frac{2n}{7} \right\rfloor$

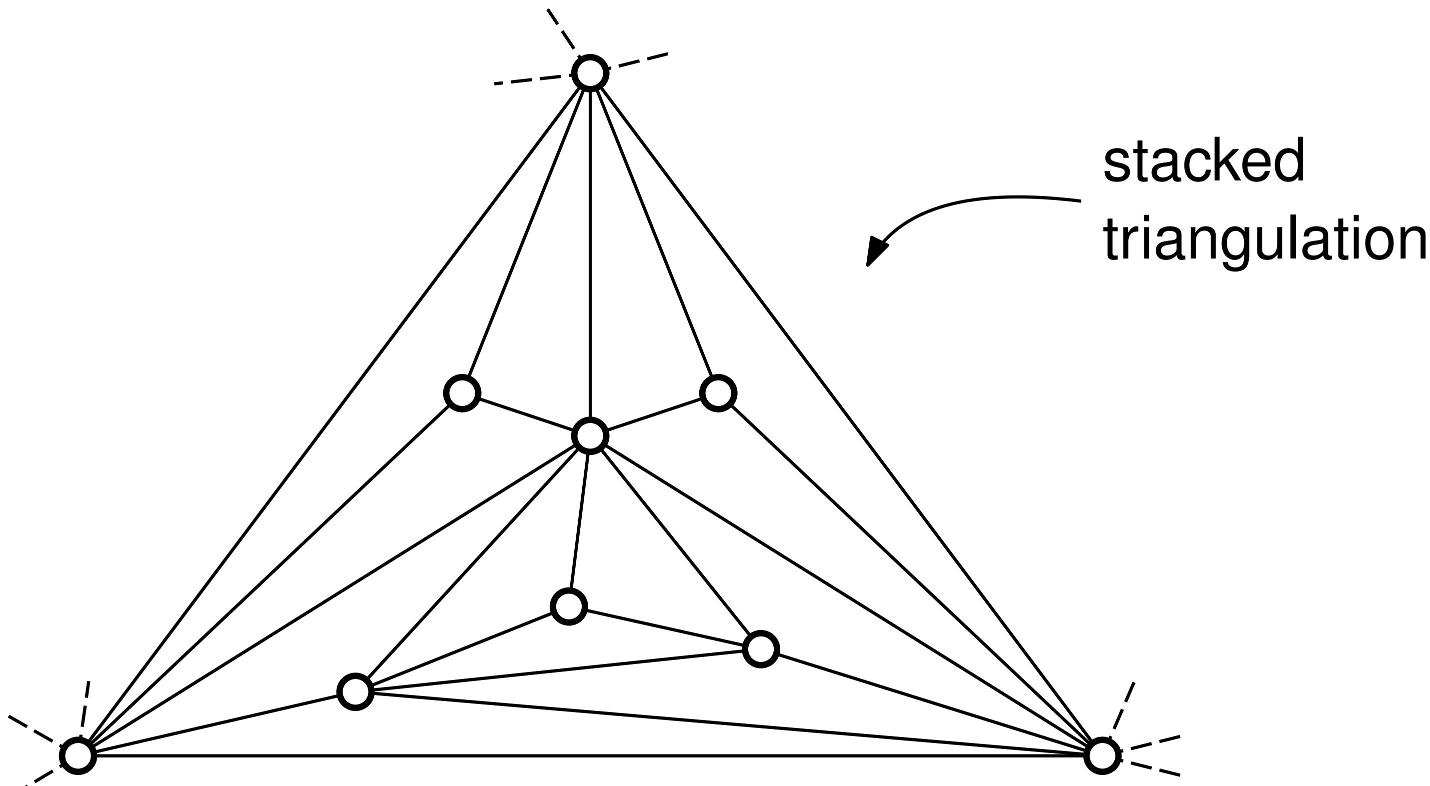
Upper Bound: Induction via Vertex Deletion

- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



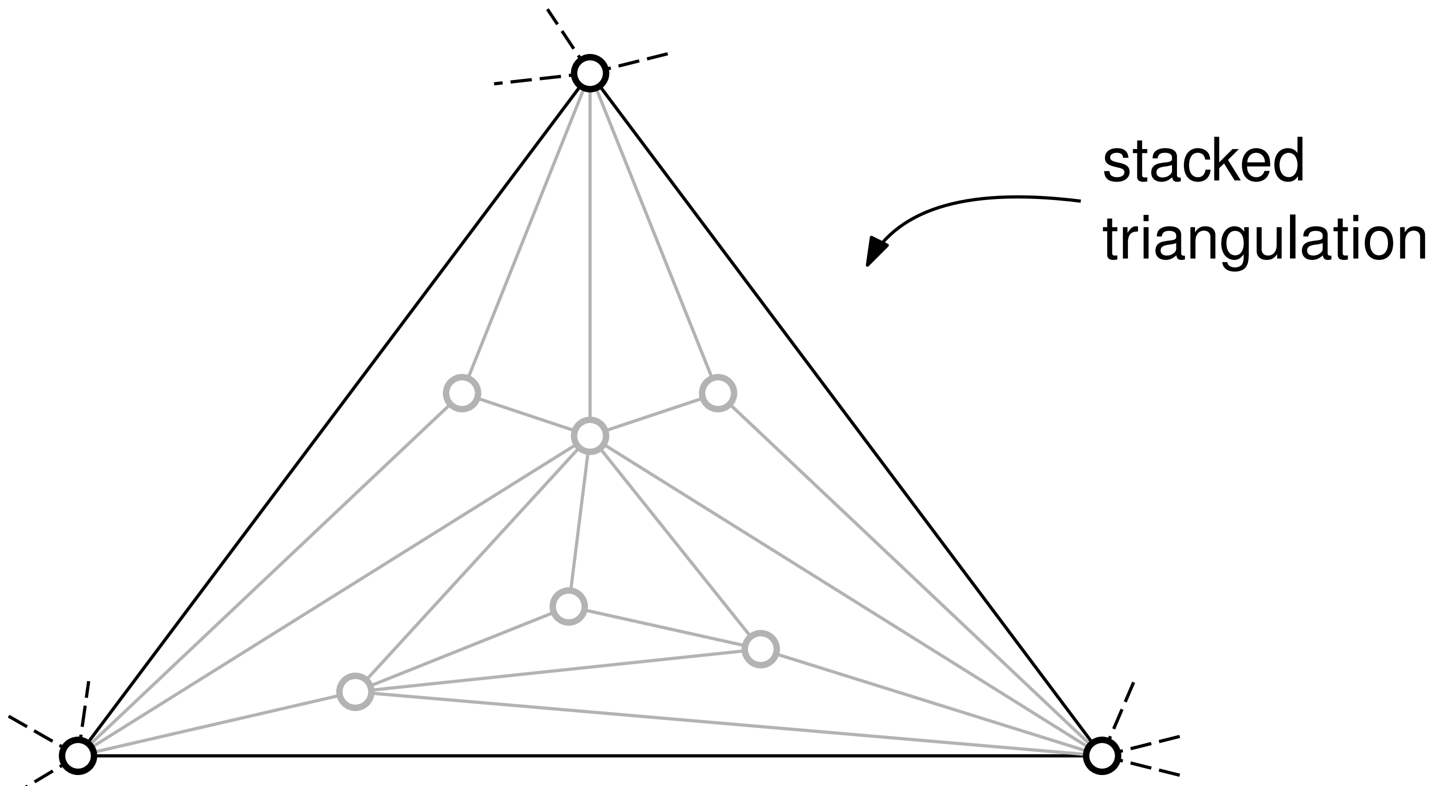
- $\frac{\ell}{k} \leq \frac{2}{7}$ in all cases $\Rightarrow$ edge guard set of size $\left\lfloor \frac{2n}{7} \right\rfloor$
- Also applied successfully for 2-degenerate quadrangulations $\left(\frac{\ell}{k} \leq \frac{1}{4}\right)$.

Induction: Examples



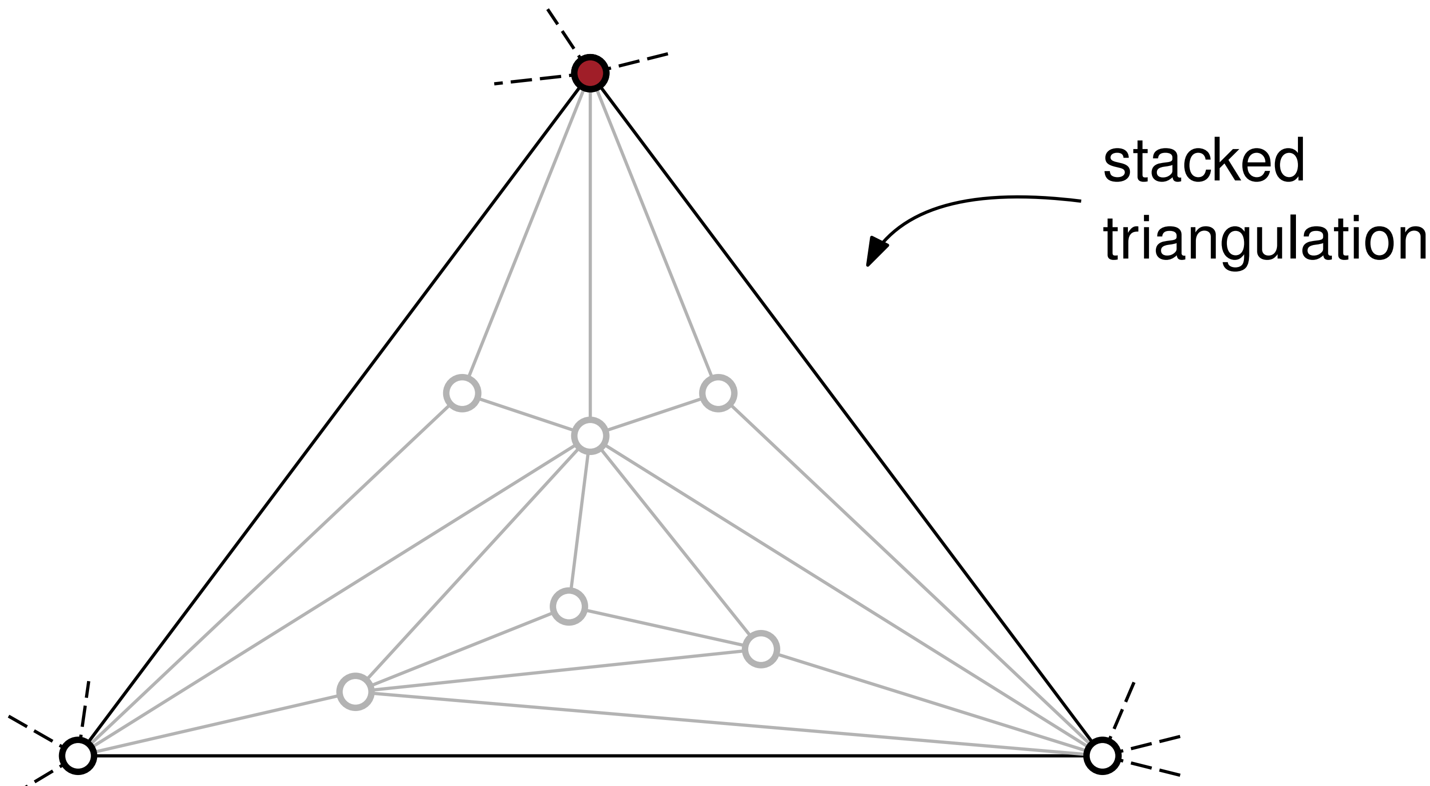
Institute of Theoretical Informatics
Algorithmics Group

Induction: Examples



- Remove inner vertices ($k = 6$).

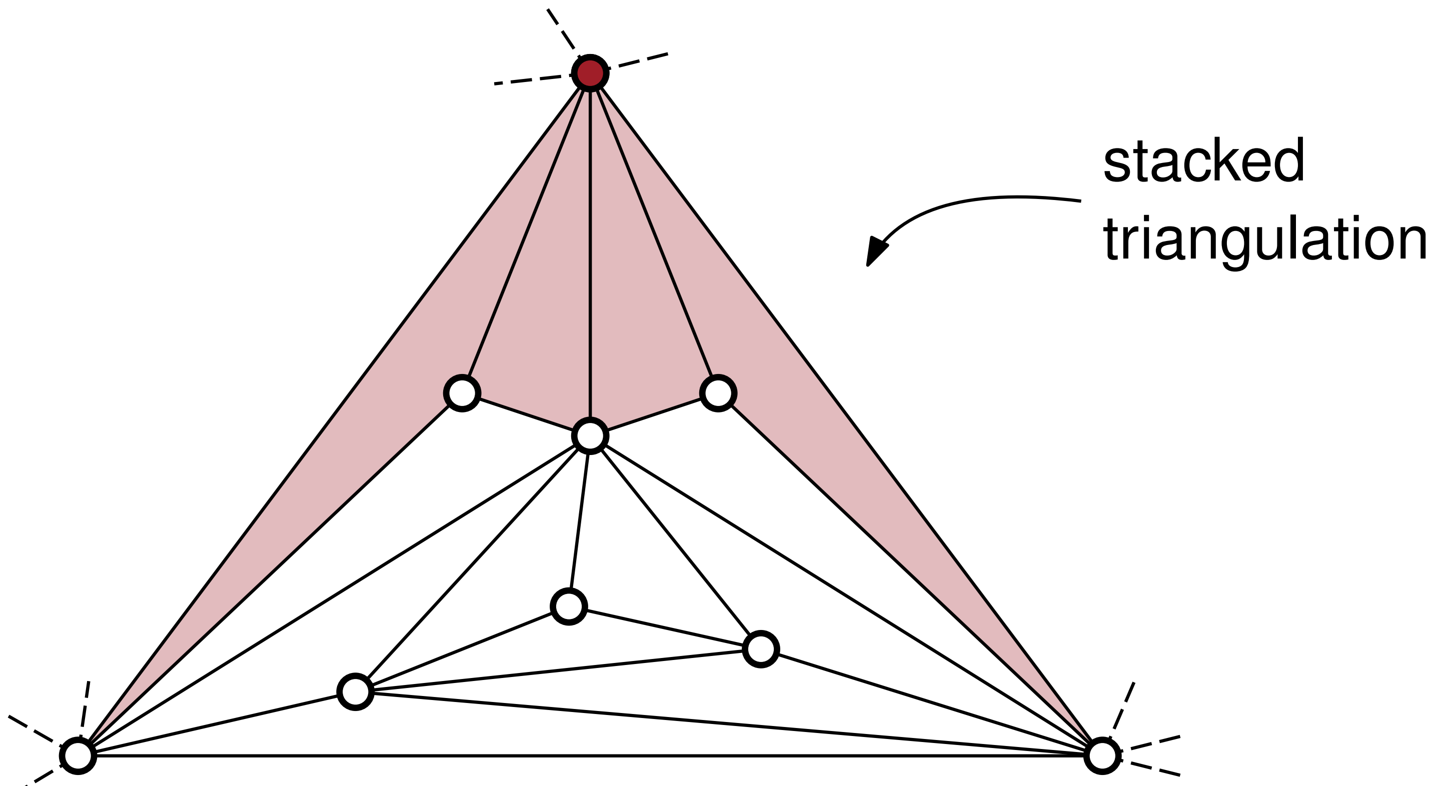
Induction: Examples



- Remove inner vertices ($k = 6$).
- Apply induction.

Induction: Examples

Karlsruhe Institute of Technology

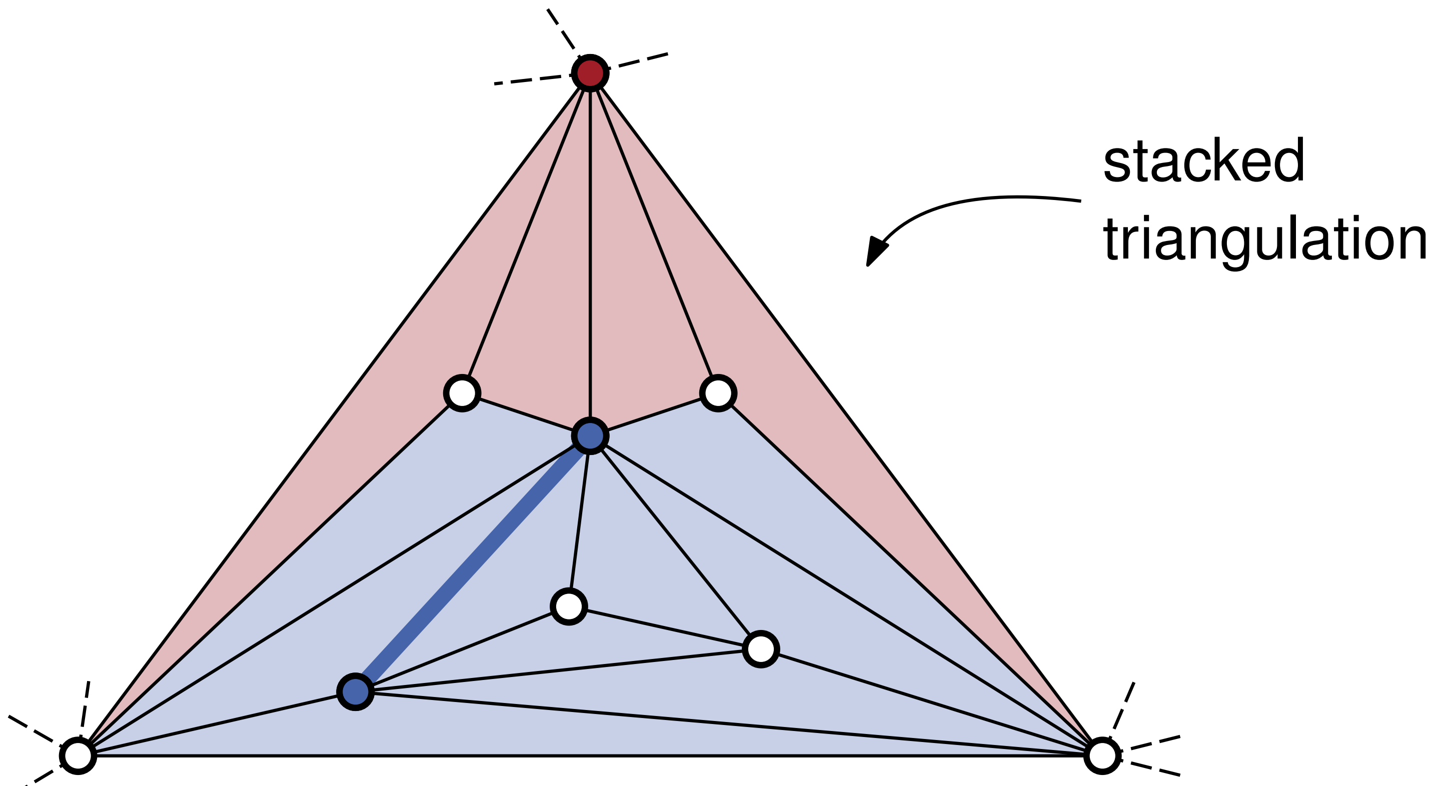


- Remove inner vertices ($k = 6$).
- Apply induction.
- Reinsert inner vertices.

Institute of Theoretical Informatics
Algorithmics Group

Induction: Examples

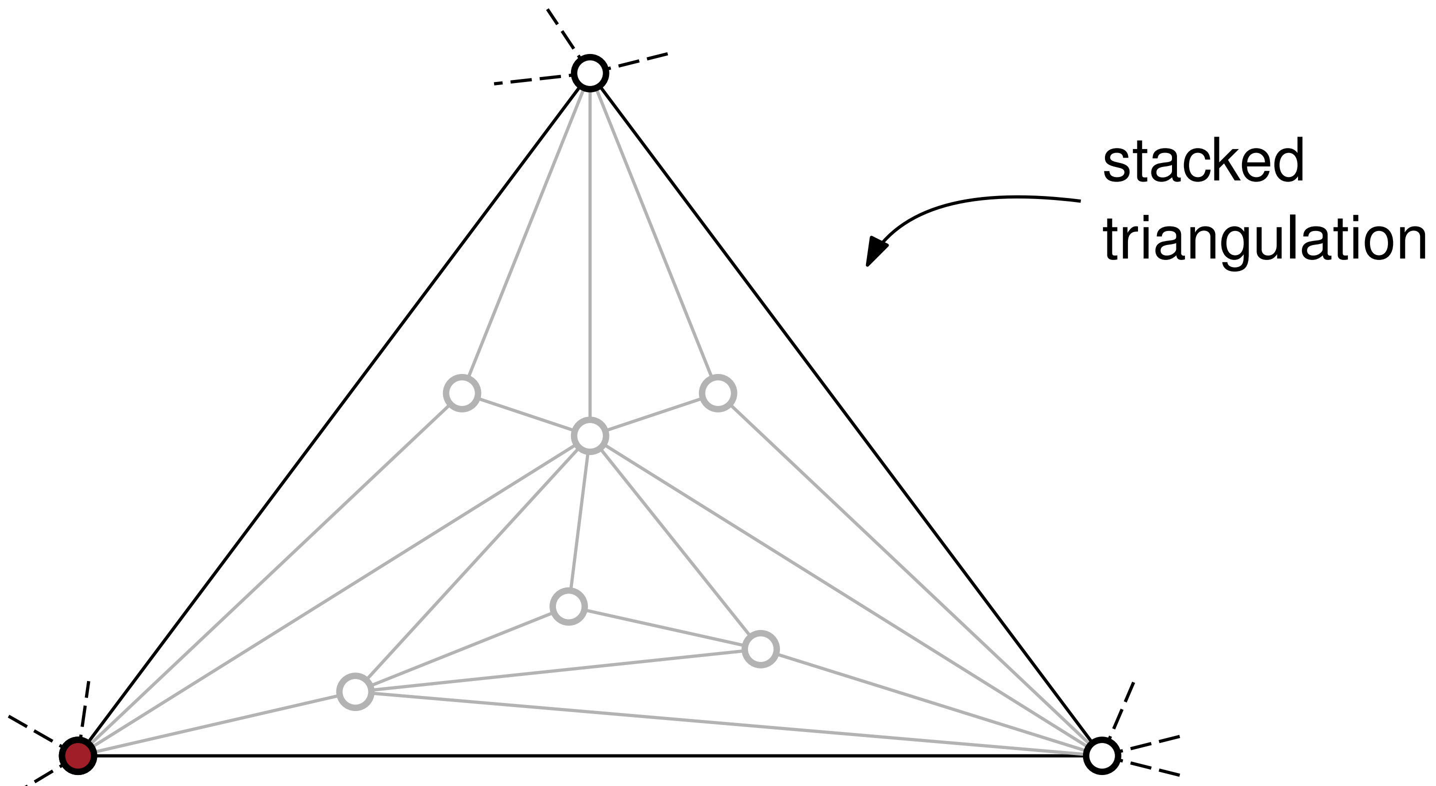
Karlsruhe Institute of Technology



- Remove inner vertices ($k = 6$).
- Apply induction.
- Reinsert inner vertices.
- Add addtional edge ($\ell = 1$), so $\frac{\ell}{k} = \frac{1}{6} \leq \frac{2}{7}$.

Institute of Theoretical Informatics
Algorithmics Group

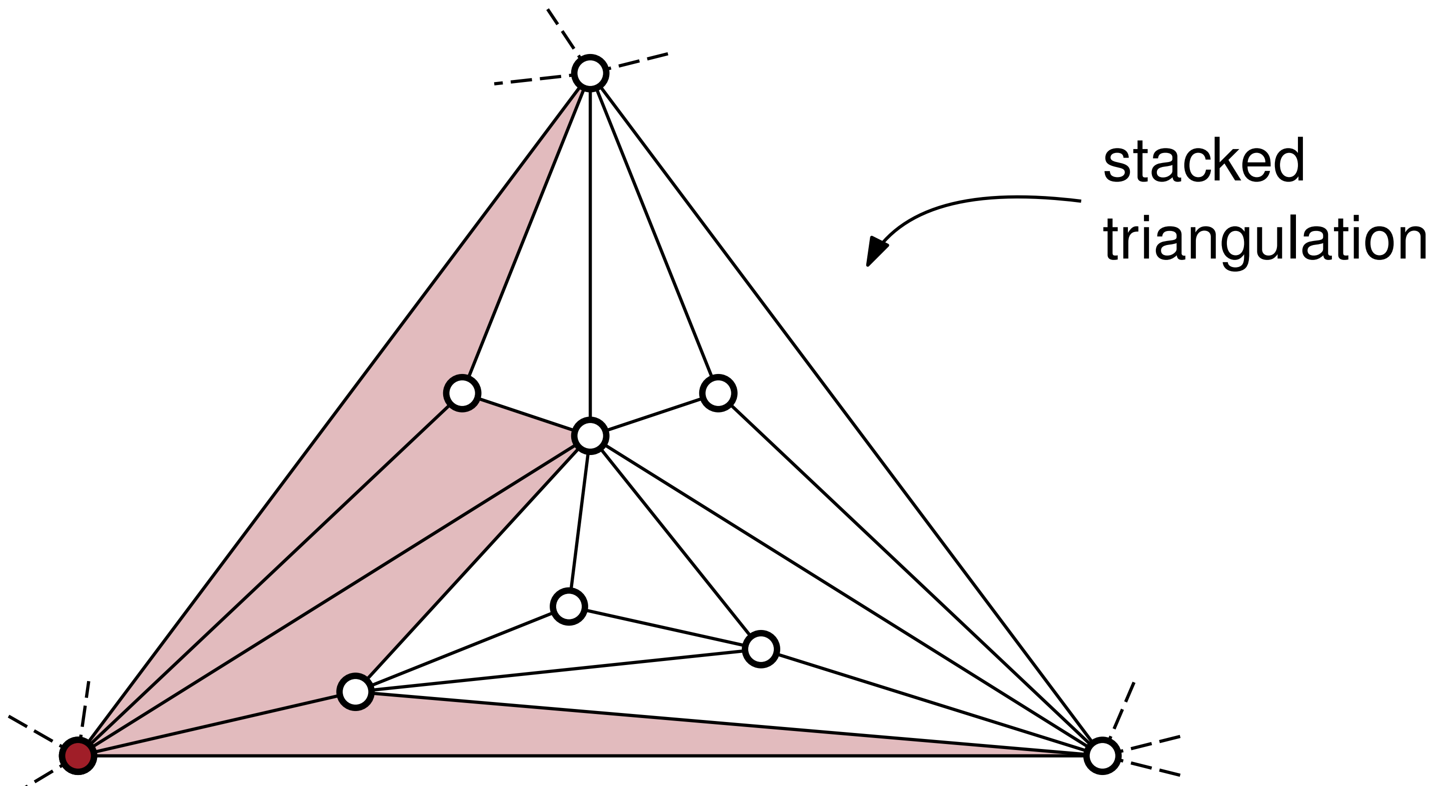
Induction: Examples



- Remove inner vertices ($k = 6$).
- Apply induction.

Institute of Theoretical Informatics
Algorithmics Group

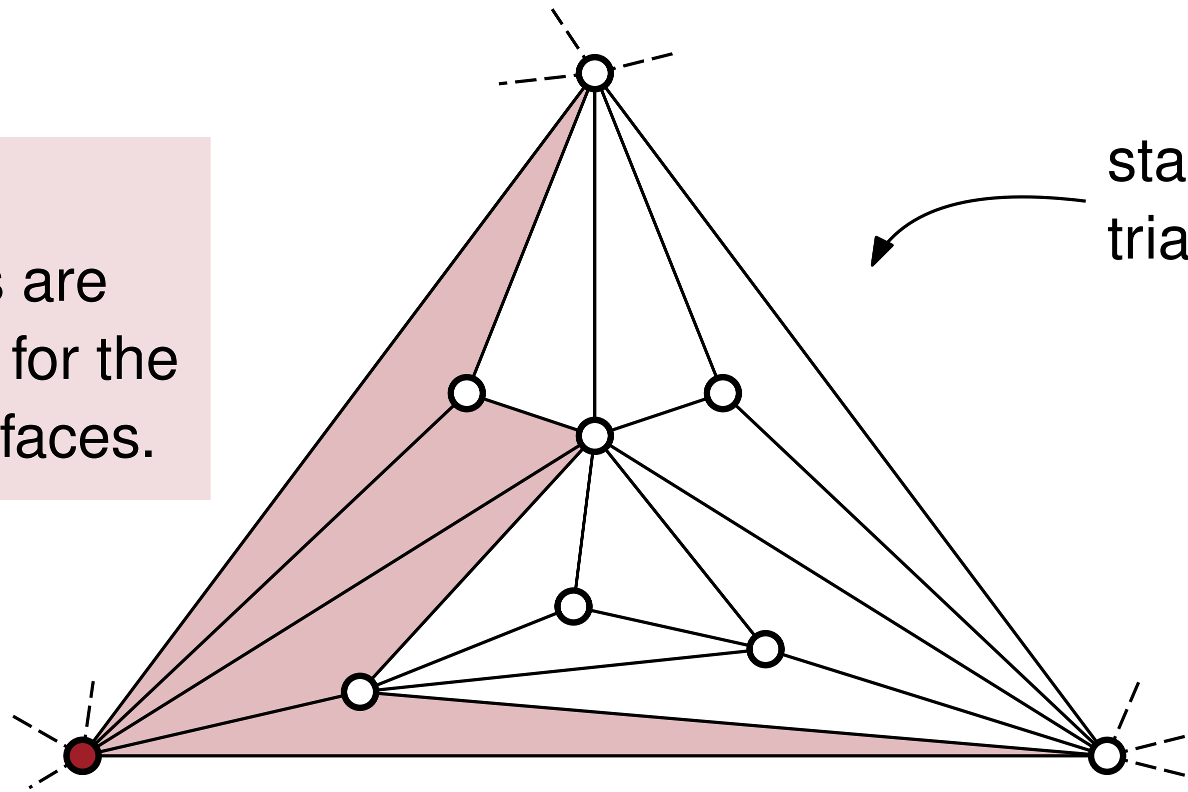
Induction: Examples



- Remove inner vertices (k = 6).
- Apply induction.
- Reinsert inner vertices.

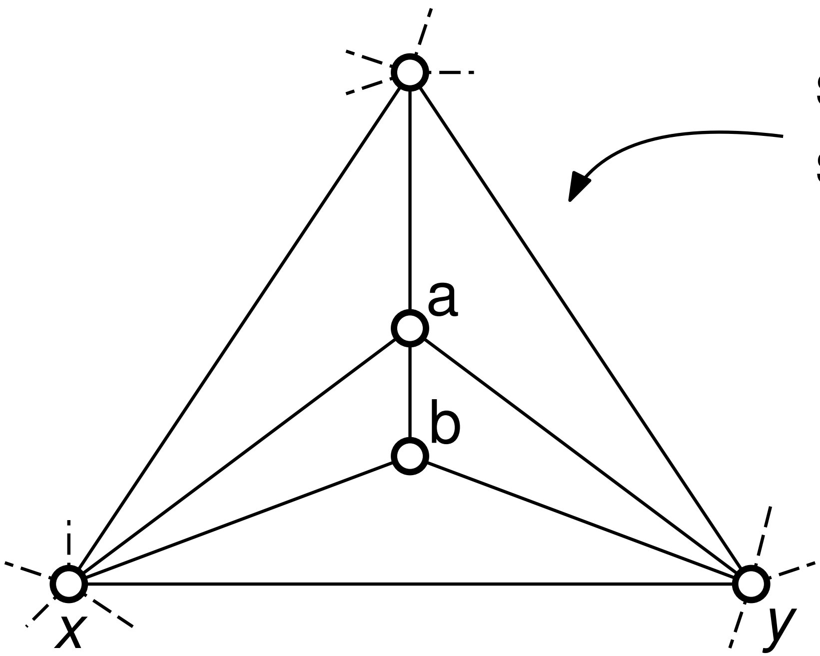
Induction: Examples

Problem:
Two edges are necessary for the remaining faces.



- Remove inner vertices (k = 6).
- Apply induction.
- Reinsert inner vertices.

Induction: Trick



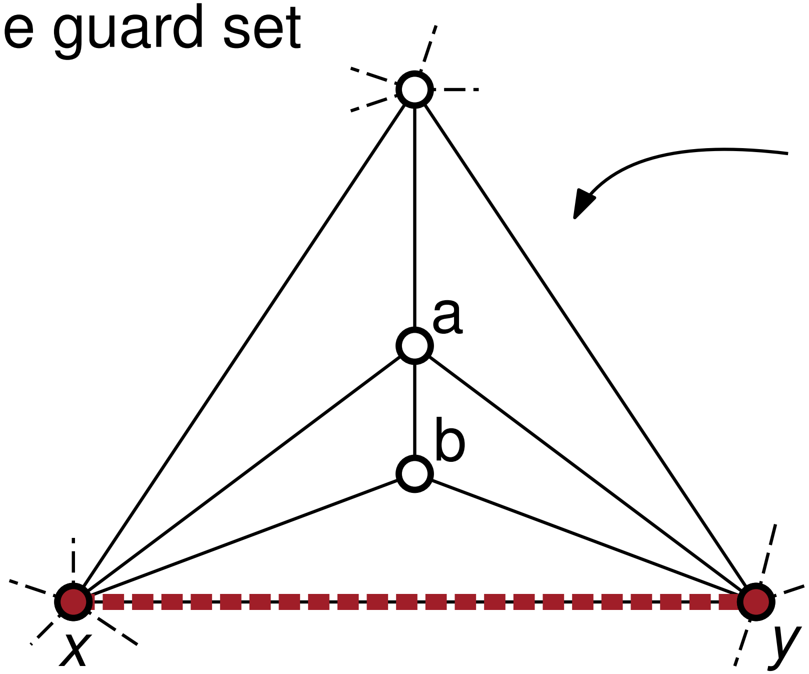
Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$



somewhere in a stacked triangulation

Lemma

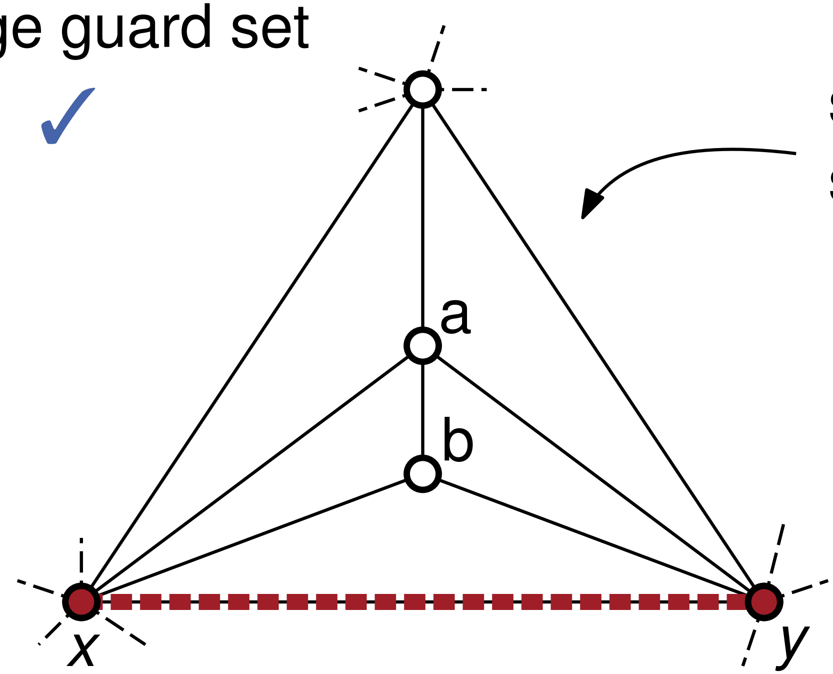
There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Institute of Theoretical Informatics
Algorithmics Group

Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓



somewhere in a stacked triangulation

Lemma

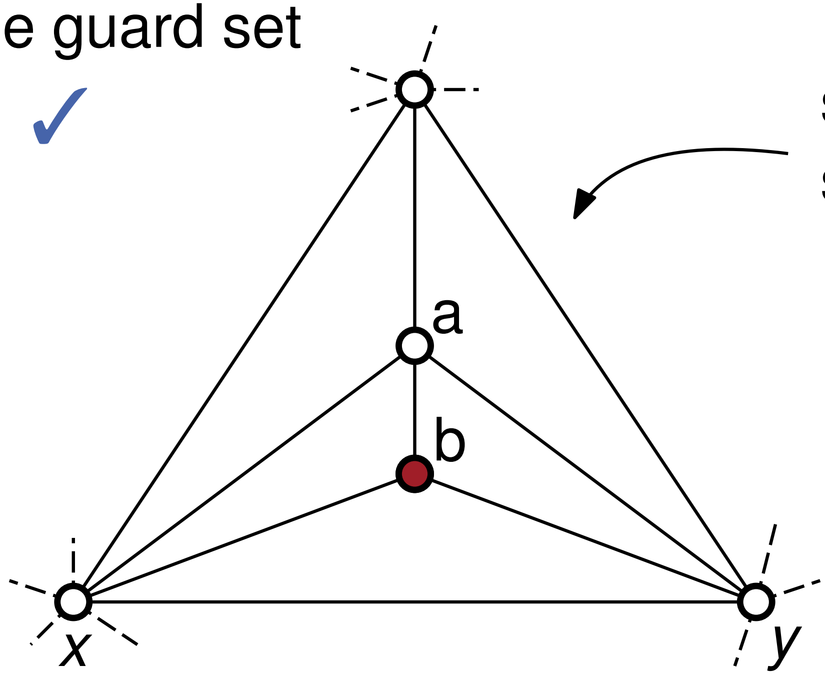
There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$



somewhere in a stacked triangulation

Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

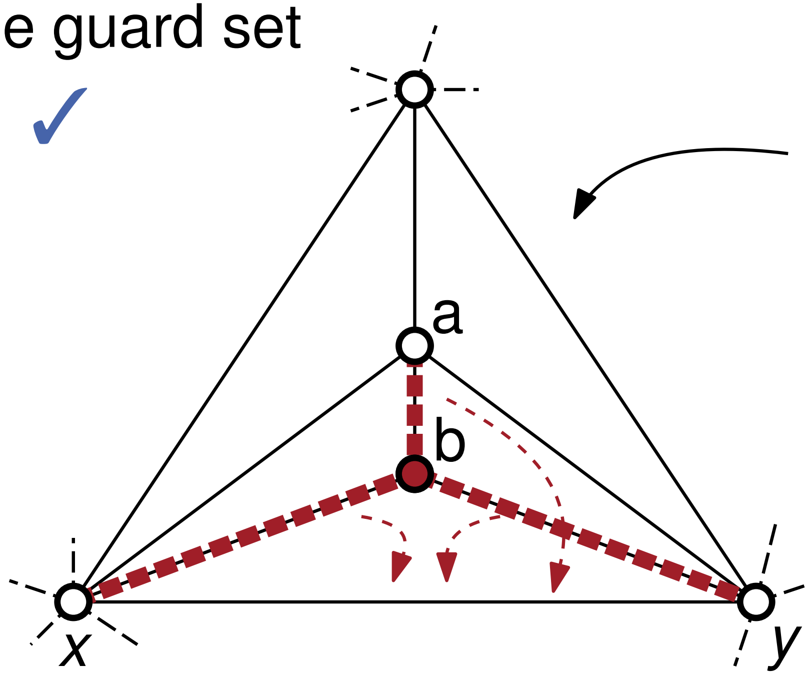
Institute of Theoretical Informatics
Algorithmics Group

Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$



somewhere in a stacked triangulation

Lemma

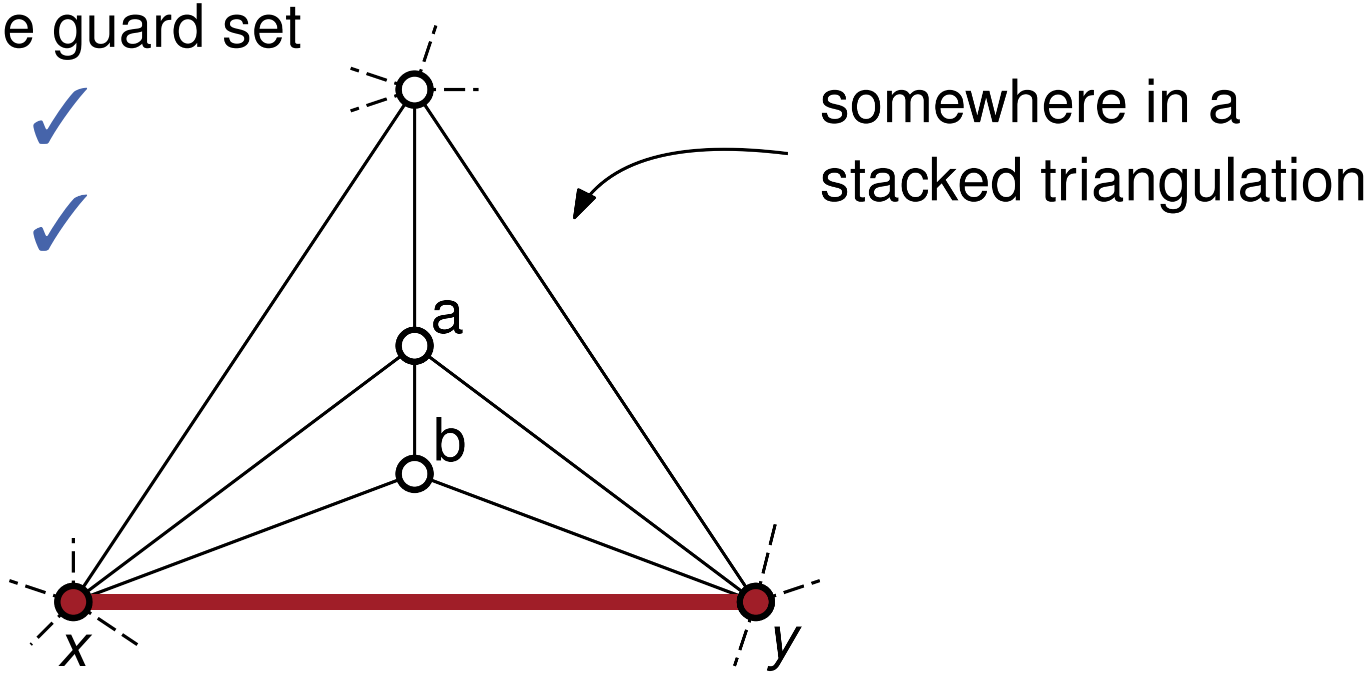
There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓



Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Institute of Theoretical Informatics
Algorithmics Group

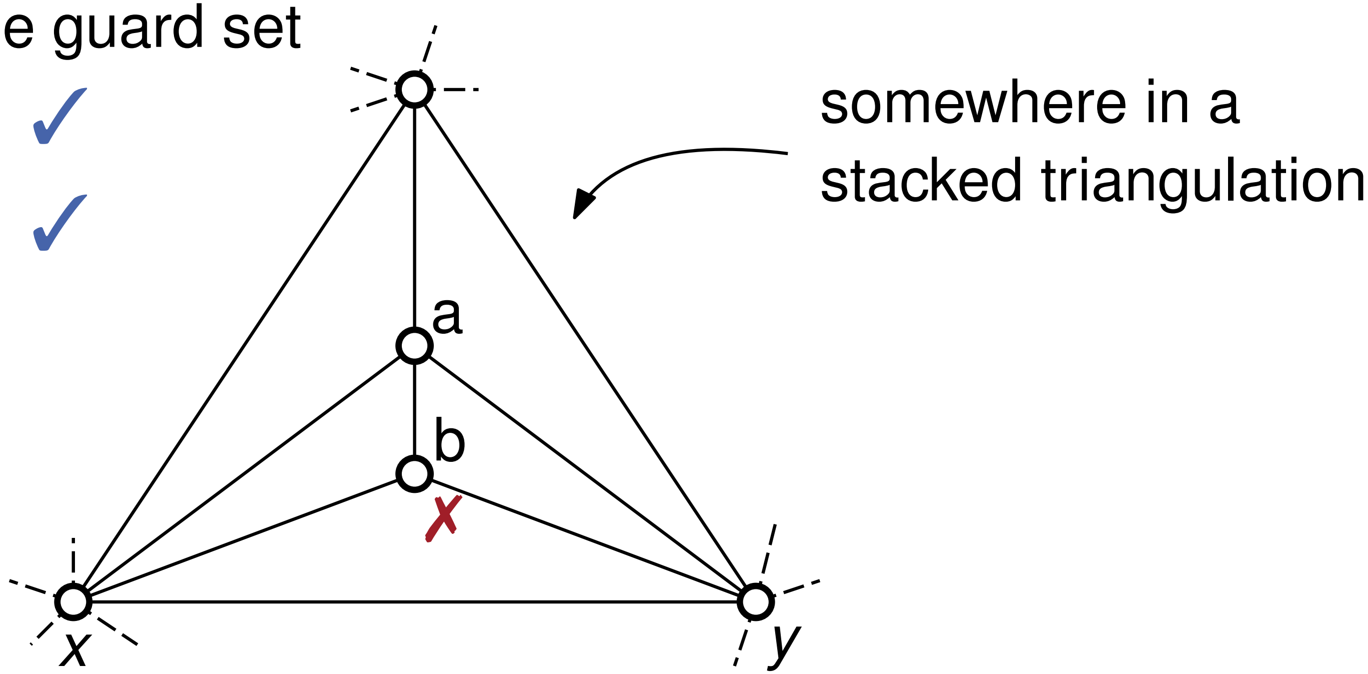
Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓

Case 2: $b \notin V(\Gamma)$



Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Induction: Trick

Karlsruhe Institute of Technology

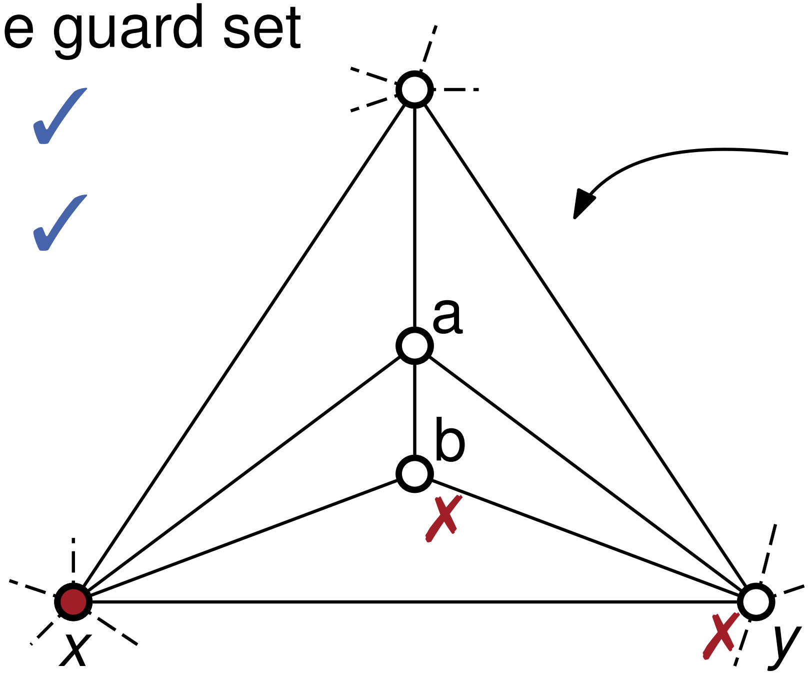
Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓

Case 2: $b \notin V(\Gamma)$

w.l.o.g. $\rightsquigarrow x \in V(\Gamma)$



somewhere in a stacked triangulation

Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Institute of Theoretical Informatics
Algorithmics Group

Induction: Trick

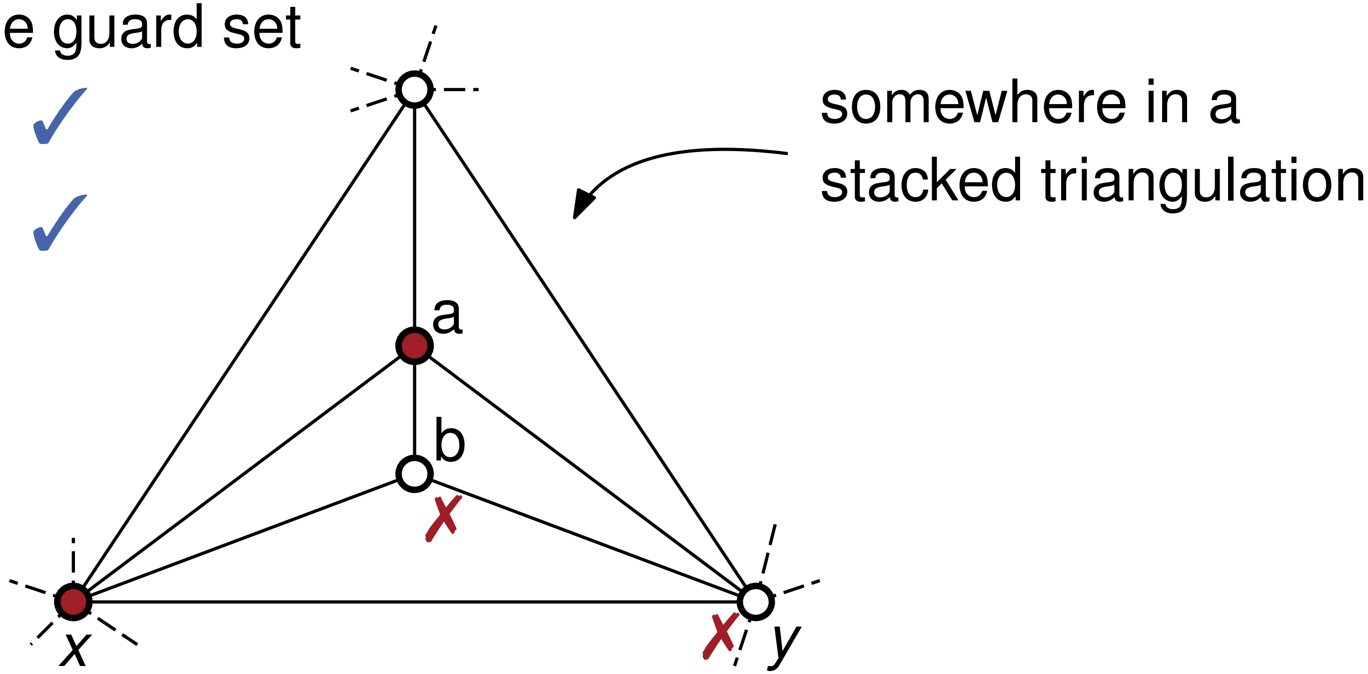
Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓

Case 2: $b \notin V(\Gamma)$

w.l.o.g. $\rightsquigarrow x \in V(\Gamma)$
$\rightsquigarrow a \in V(\Gamma)$



Lemma
There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Induction: Trick

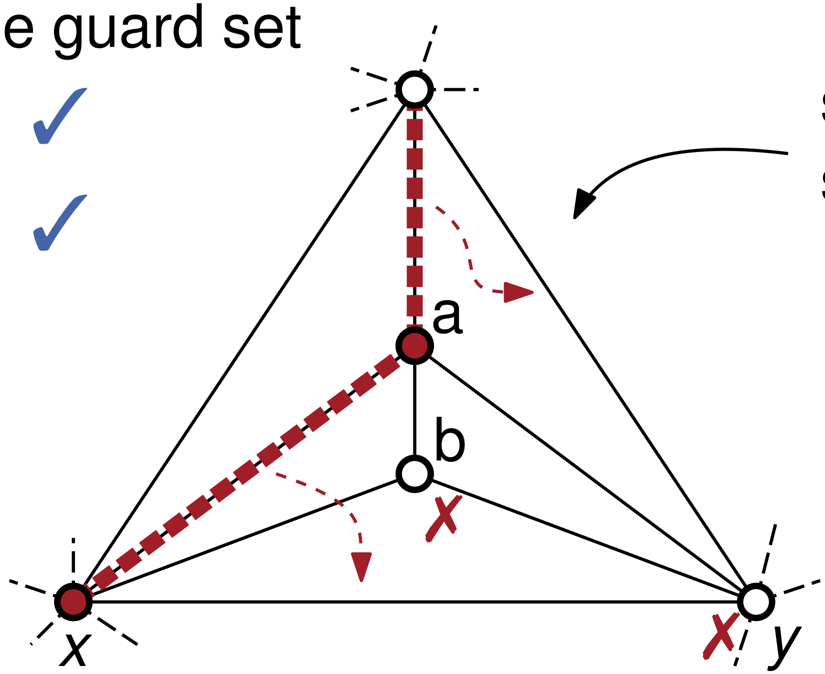
Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓

Case 2: $b \notin V(\Gamma)$

w.l.o.g. $\rightsquigarrow x \in V(\Gamma)$
$\rightsquigarrow a \in V(\Gamma)$



Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

Institute of Theoretical Informatics
Algorithmics Group

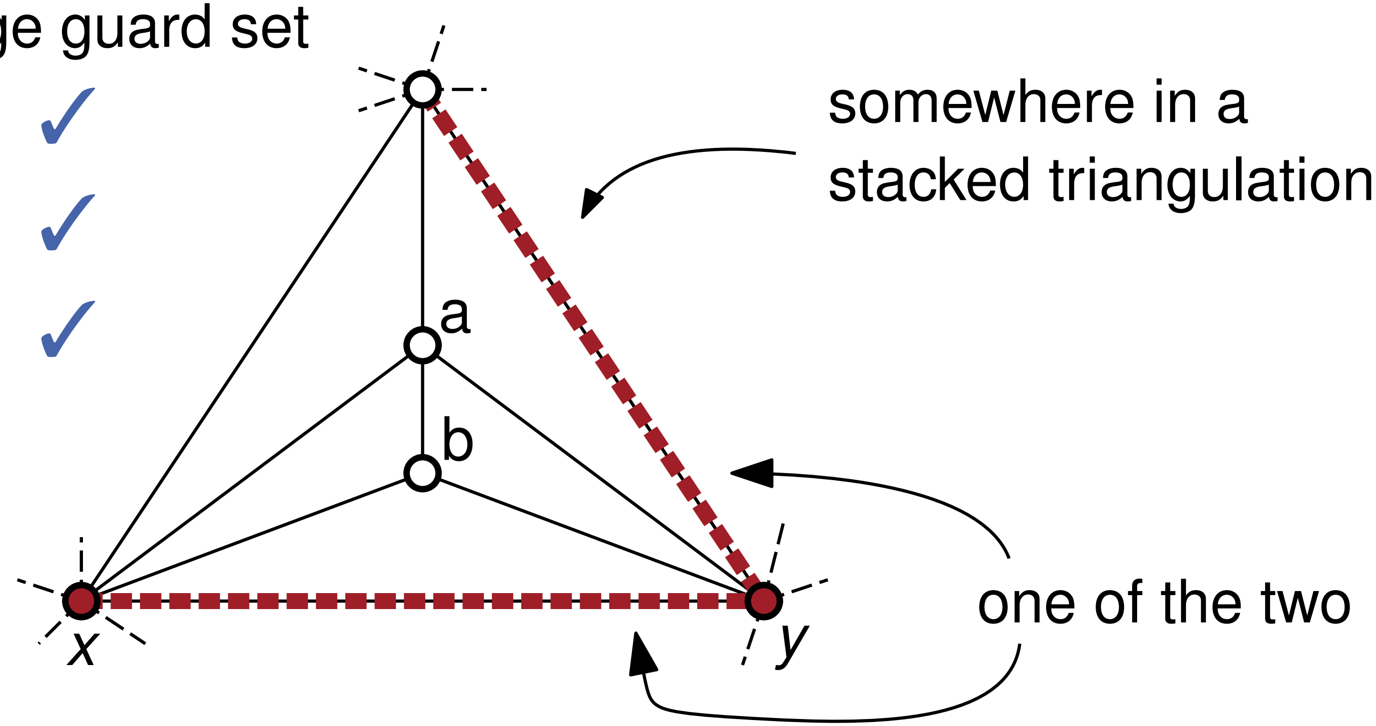
Induction: Trick

Γ : minimum size edge guard set

Case 0: $x, y \in V(\Gamma)$ ✓

Case 1: $b \in V(\Gamma)$ ✓

Case 2: $b \notin V(\Gamma)$ ✓

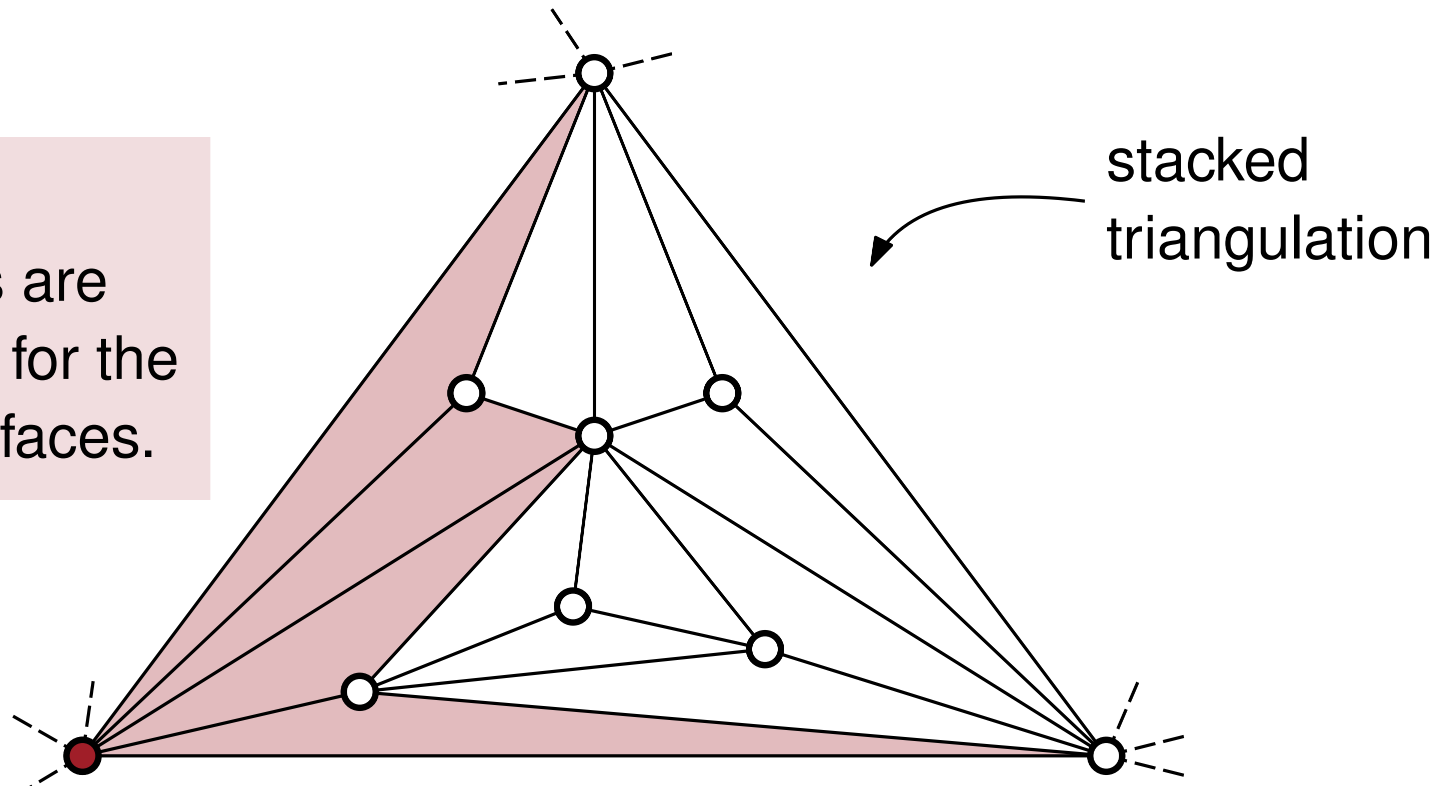


Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

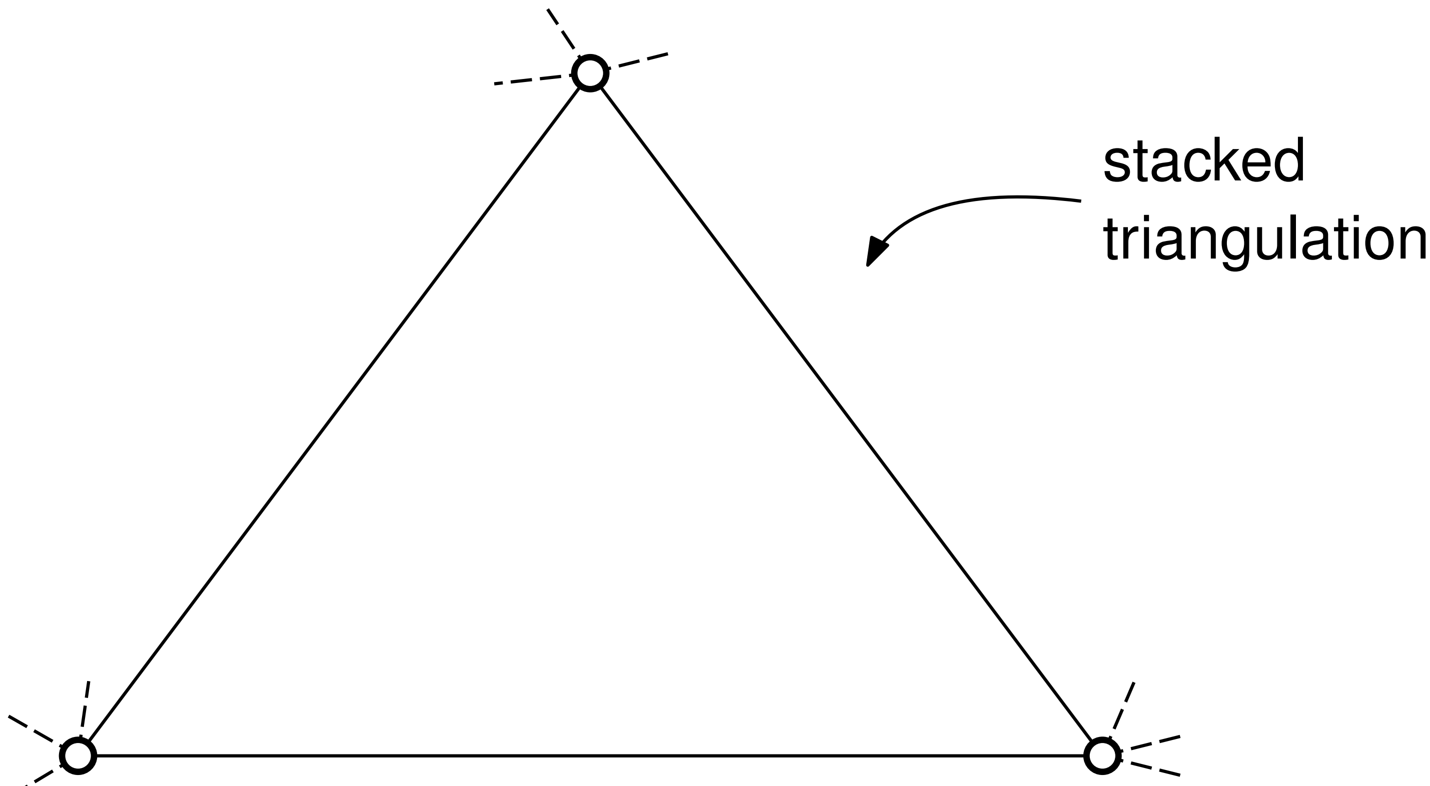
Induction Example: Revisited

Problem:
Two edges are necessary for the remaining faces.



- Remove inner vertices ($k = 6$).
- Apply induction.
- Reinsert inner vertices.

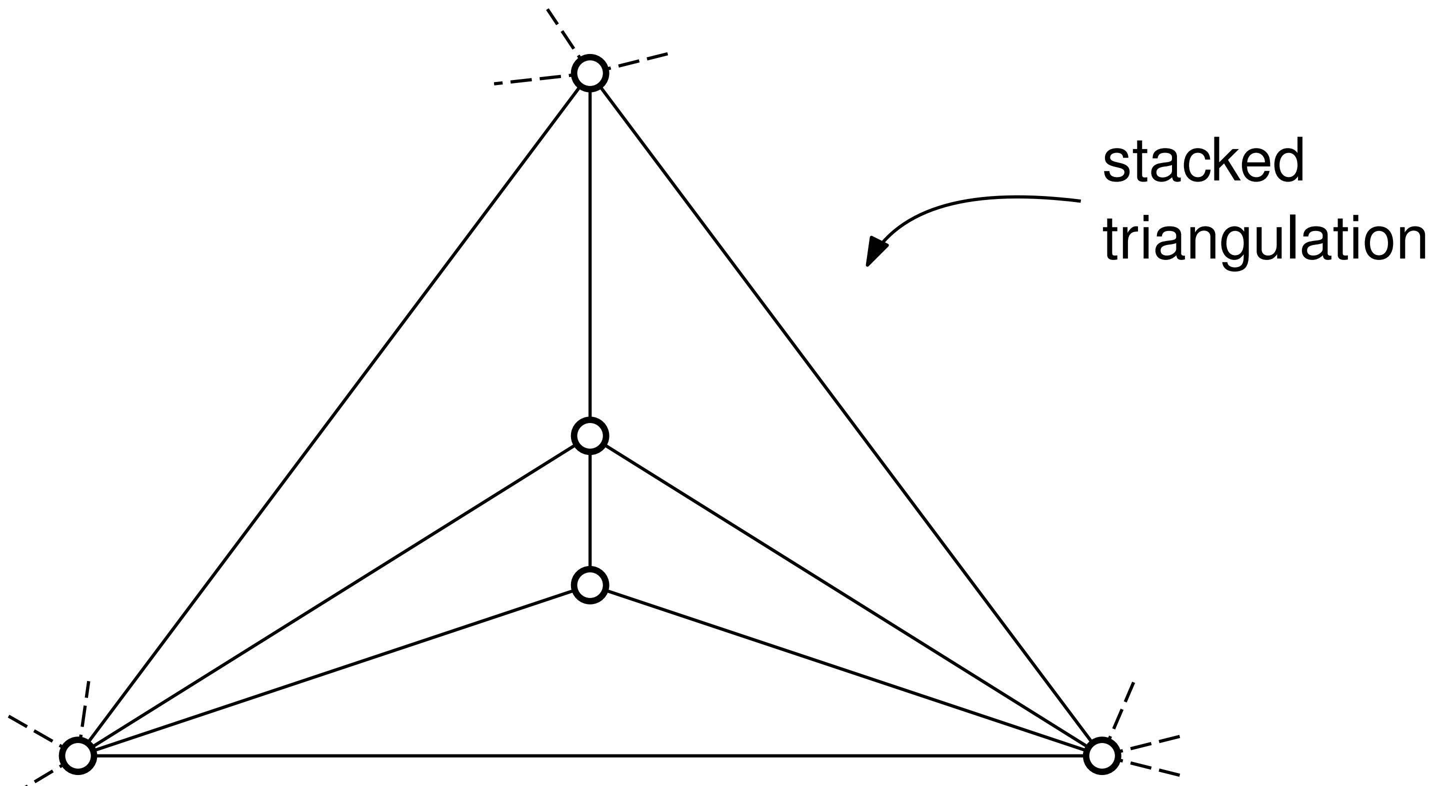
Induction Example: Revisited



- Remove inner vertices ($k = 6$).

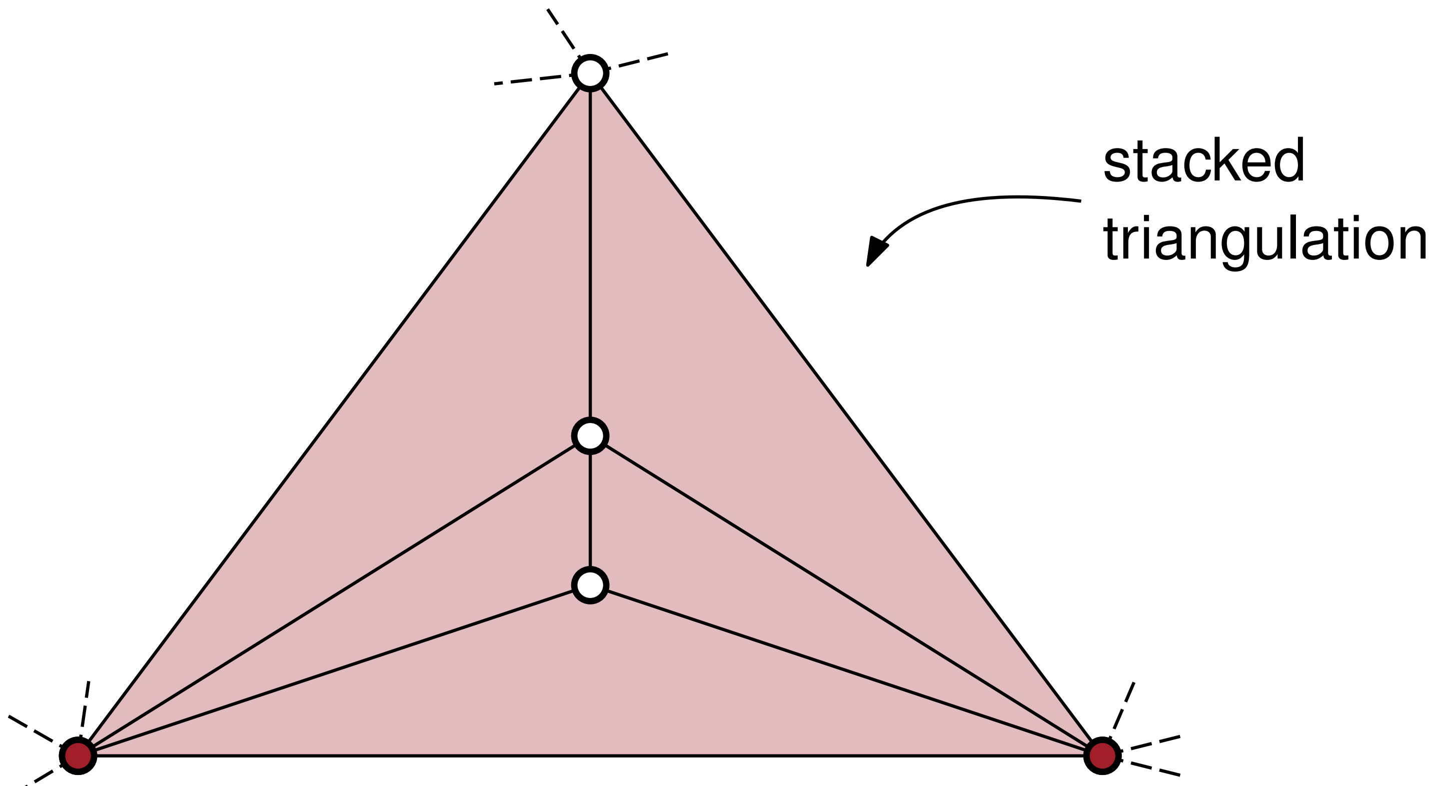
Institute of Theoretical Informatics
Algorithmics Group

Induction Example: Revisited



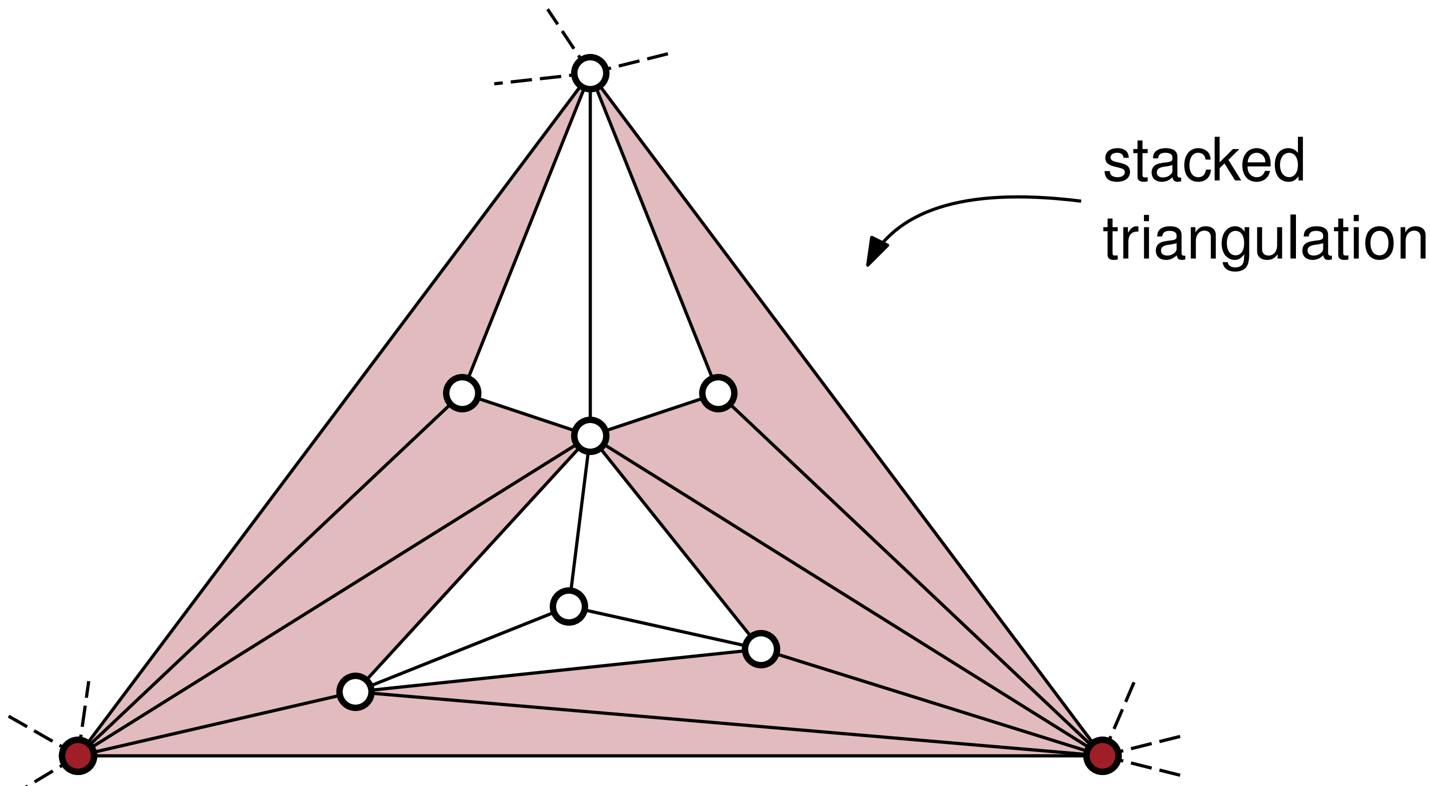
- Remove inner vertices ($k = 6$).
- Add two new vertices ($k = 6 \rightsquigarrow k = 4$).

Induction Example: Revisited



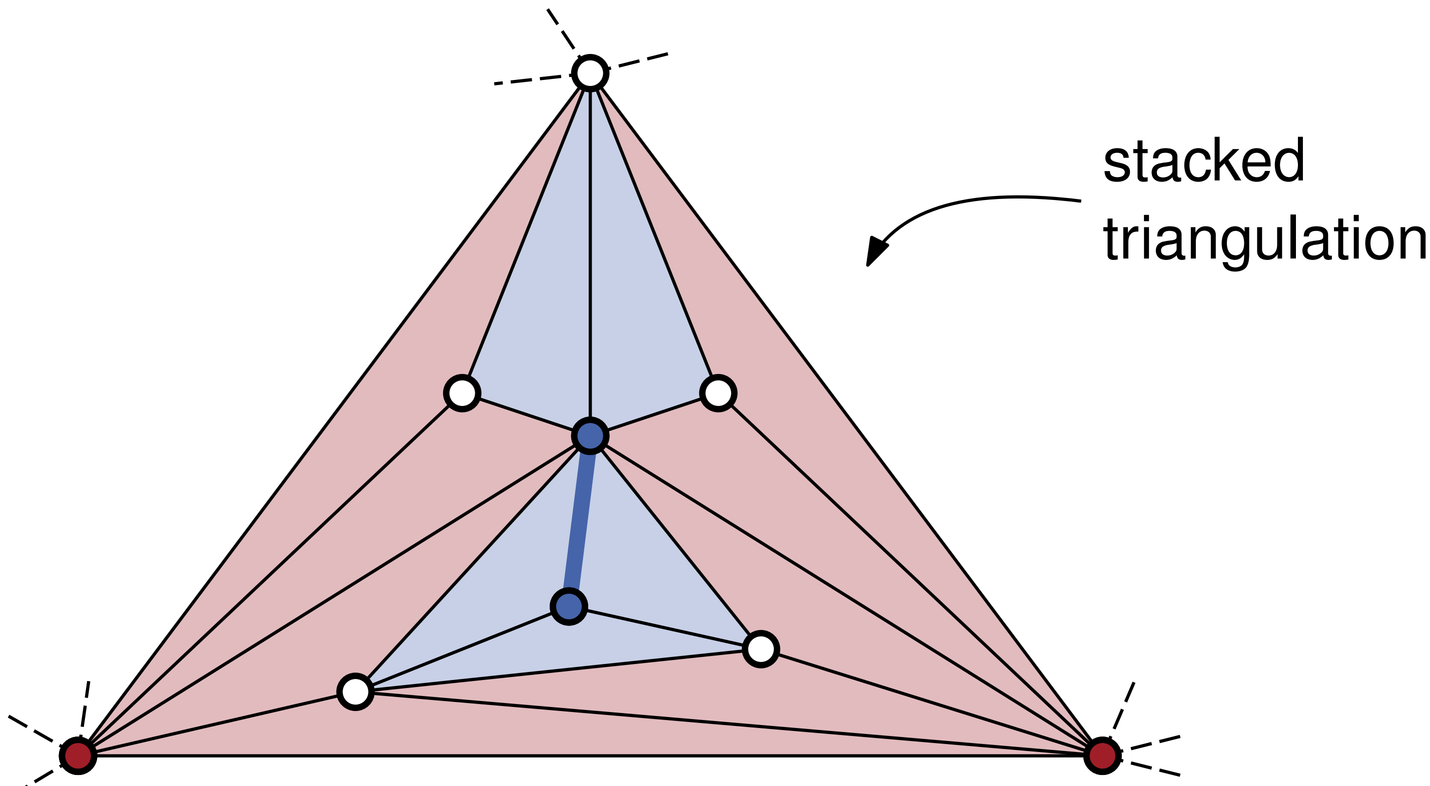
- Remove inner vertices ($k = 6$).
- Add two new vertices ($k = 6 \rightsquigarrow k = 4$).
- Apply lemma from last slide.

Induction Example: Revisited



- Remove inner vertices ($k = 6$).
- Add two new vertices ($k = 6 \leadsto k = 4$).
- Apply lemma from last slide.
- Reinsert old vertices.

Induction Example: Revisited



- Remove inner vertices ($k = 6$).
- Add two new vertices ($k = 6 \rightsquigarrow k = 4$).
- Apply lemma from last slide.
- Reinsert old vertices. One more edge suffices ($\ell = 1$), so $\frac{\ell}{k} = \frac{1}{4} \leq \frac{2}{7}$.

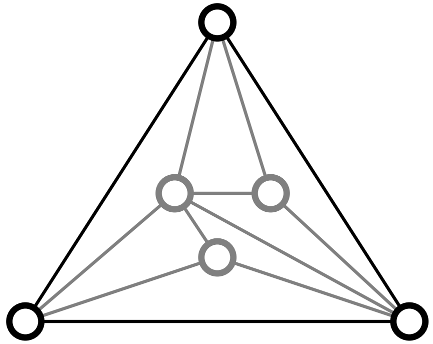
Addtitional Challenges

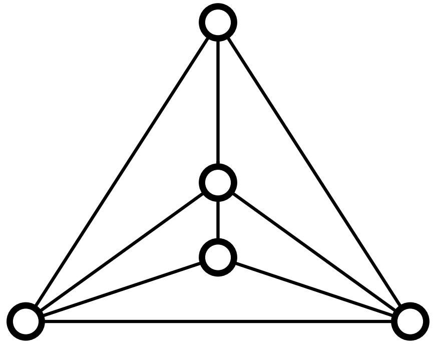
What are the key steps in the proof?

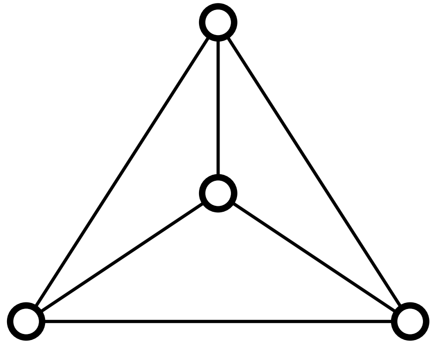
Which *k* vertices to remove?

Three different ”tricks”.

Grouping of different hundreds of configurations.





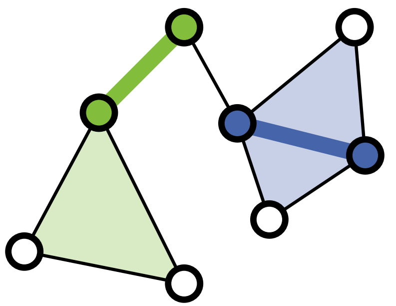


Open Problems

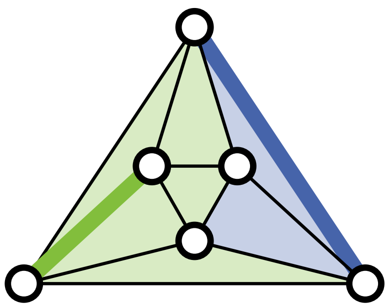
Karlsruhe Institute of Technology

How many edge guards are sometimes necessary and always sufficient for...

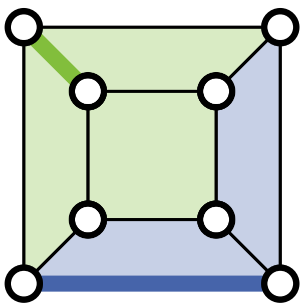
...plane graphs?



...triangulations?
(4-connected)



...quadrangulations?

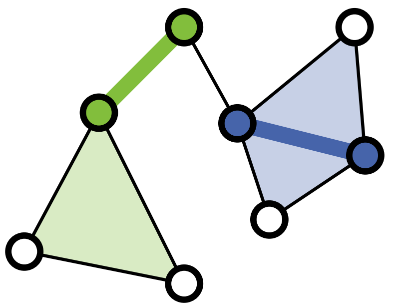


Institute of Theoretical Informatics
Algorithmics Group

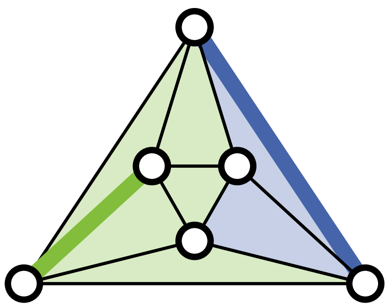
Open Problems

How many edge guards are sometimes necessary and always sufficient for...

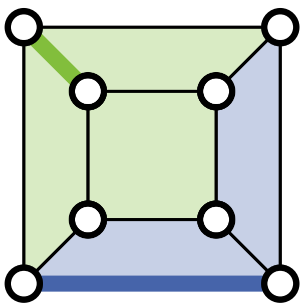
...plane graphs?



...triangulations?
(4-connected)



...quadrangulations?



Thank you! **Questions?**