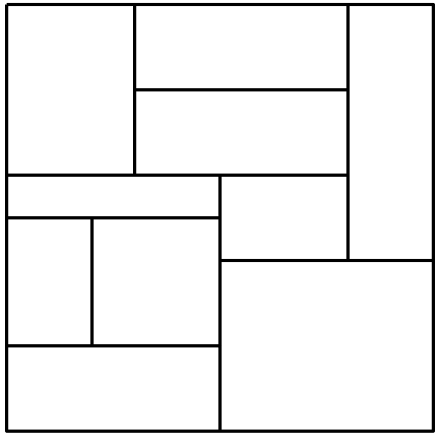
Karlsruhe Institute of Technology

The Air-Pressure Method for Area-Universality

Mittagsseminar · 5th October 2022
Paul Jungeblut

Floorplans

Floorplan:



Partition of a rectangle
into smaller rectangles.

Floorplans

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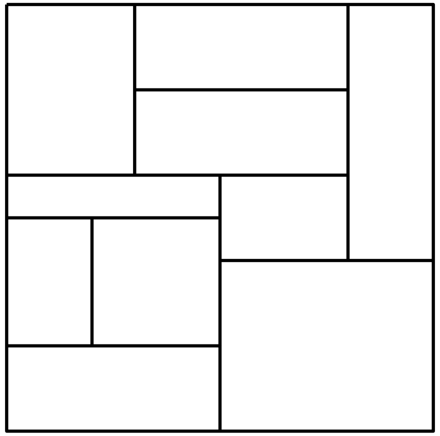

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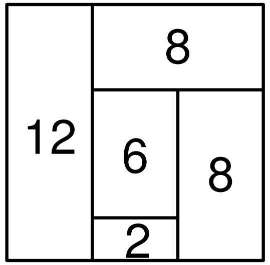
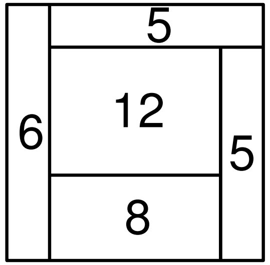
Area Universality





Definition: A floorplan is **area-universal** if for every area assignment there is an “equivalent” floorplan realizing the desired areas.

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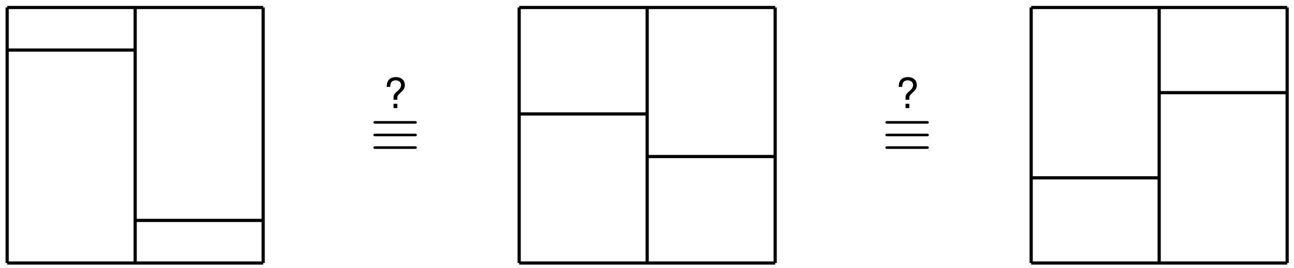
Equivalence

Question: Which floorplans should be considered equivalent?



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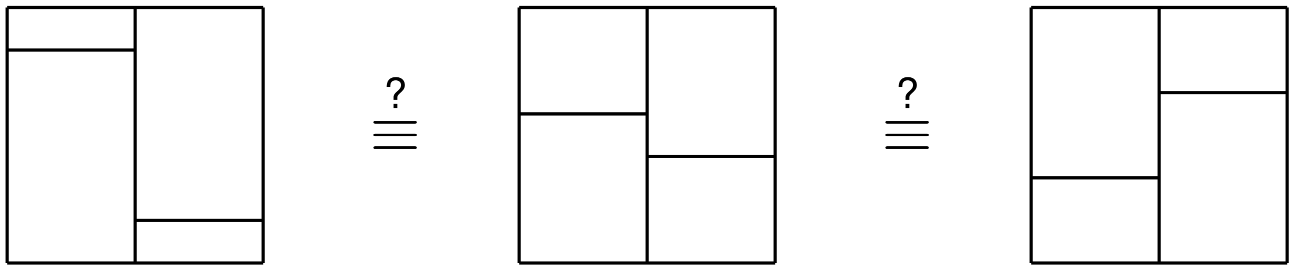

Weak Equivalence: Same segment contact graph.

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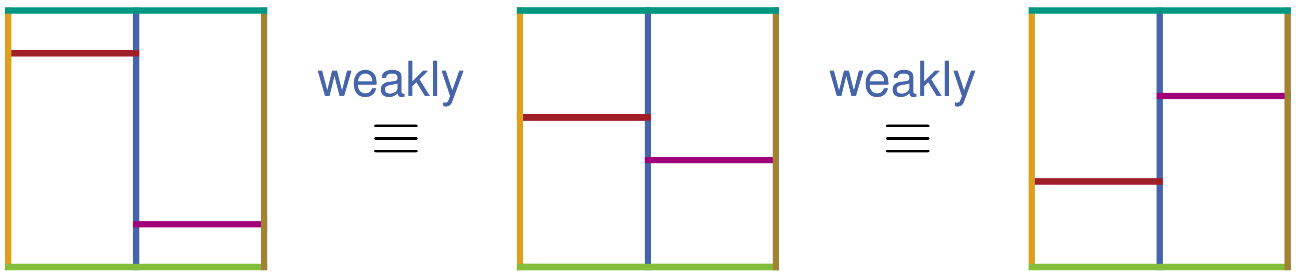

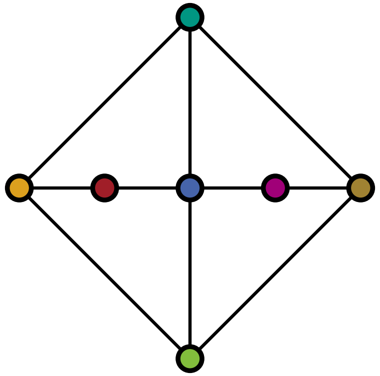

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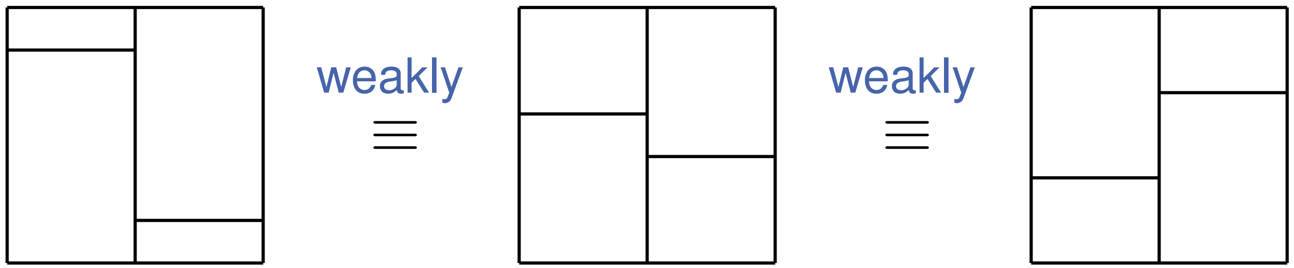

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Strong Equivalence: Same rectangle contact graph.

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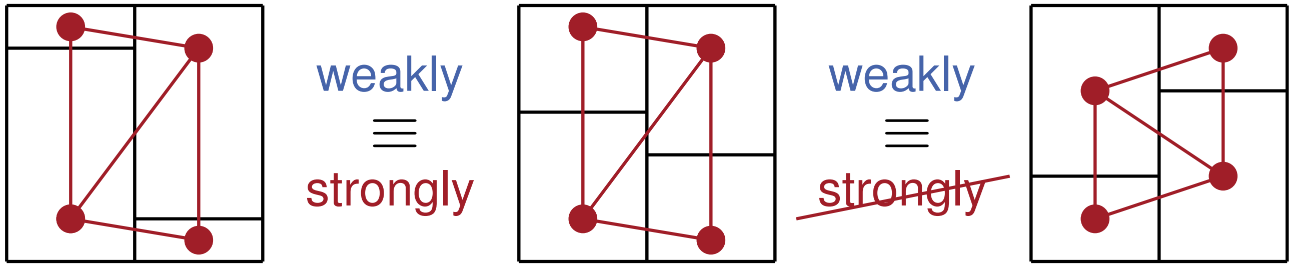

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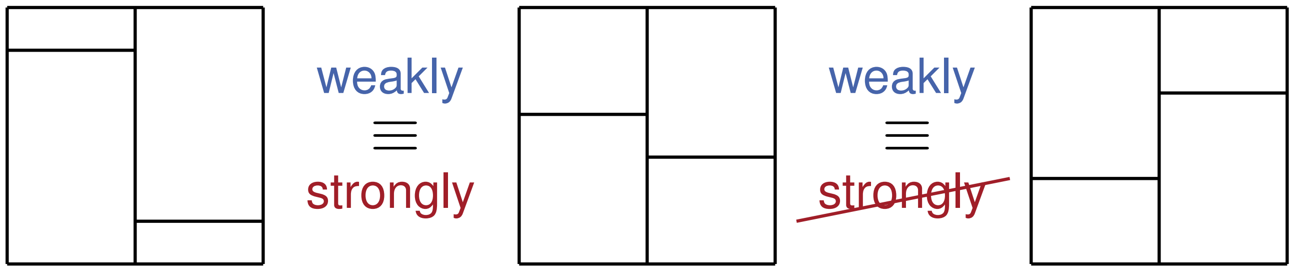

Theorem:
Floorplans are area-universal under weak equivalence.

Wimer, Koren, Cederbaum 1988
Eppstein, Mumford, Speckmann, Verbeek 2012
Felsner 2013

Weak Equivalence: Same segment contact graph.

Strong Equivalence: Same rectangle contact graph.

Air-Pressure Method

Idea: Each rectangle holds an amount of air proportional to its desired area.

- too small $\Rightarrow$ high pressure (> 1)
- too large $\Rightarrow$ low pressure (< 1)

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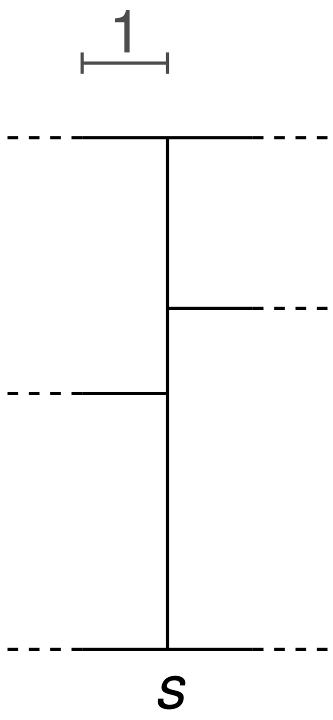
Formally:

- $w : R \to \mathbb{R}_{>0}$ — area assignment for each rectangle $r \in R$
- $p(r) = \frac{w(r)}{\text{area}(r)}$ — **pressure** of a rectangle
- $\mathcal{F}(s, r) = p(r) \cdot \ell(s, r)$ — **force** that r exerts on segment s (where $\ell(s, r)$ is length of r on s)

$$\mathcal{F}(s) = \sum_{r \text{ left of } s} \mathcal{F}(s, r) - \sum_{r \text{ right of } s} \mathcal{F}(s, r)$$

force on a vertical segment

Example



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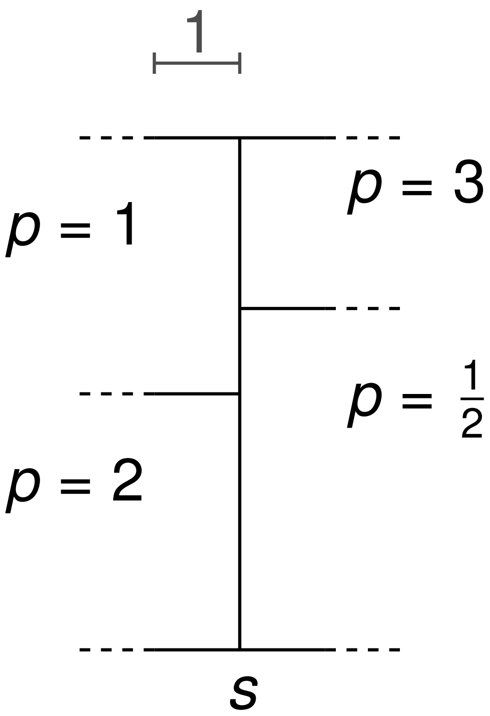

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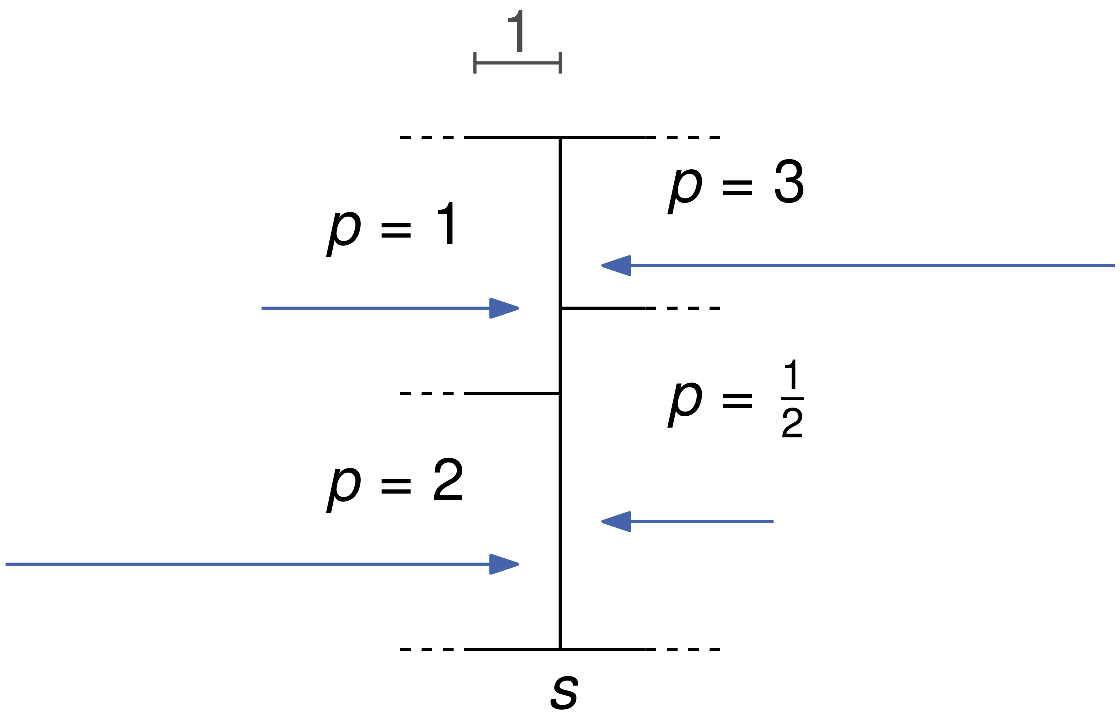

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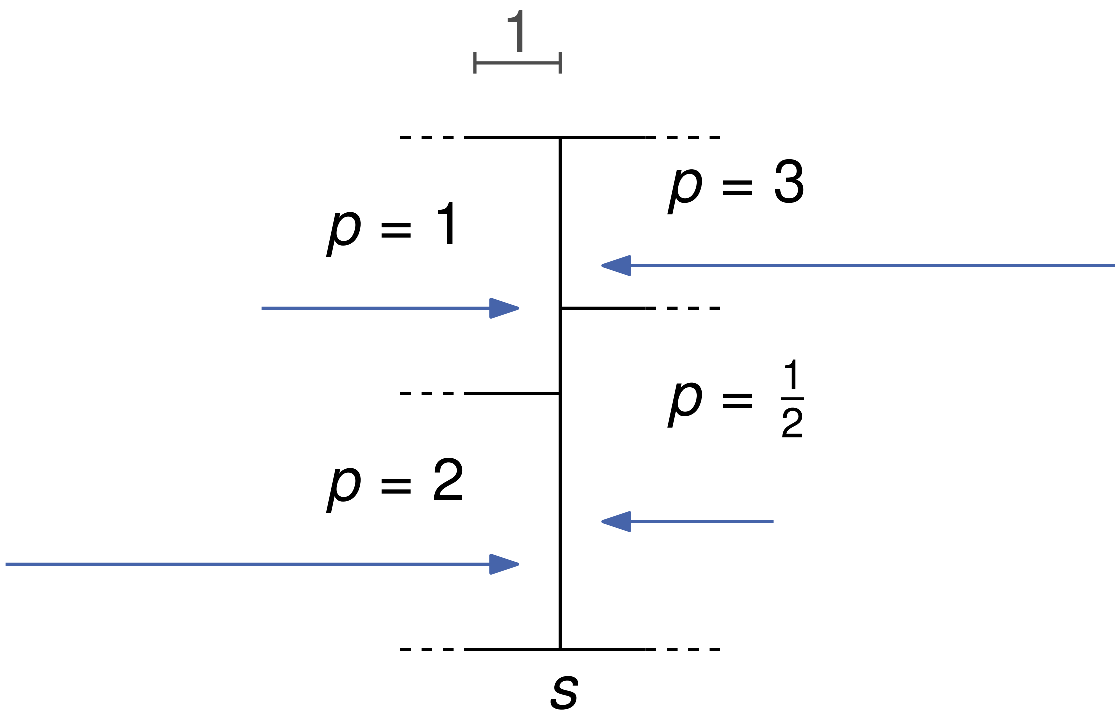

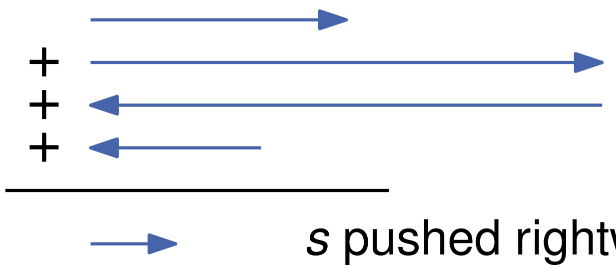

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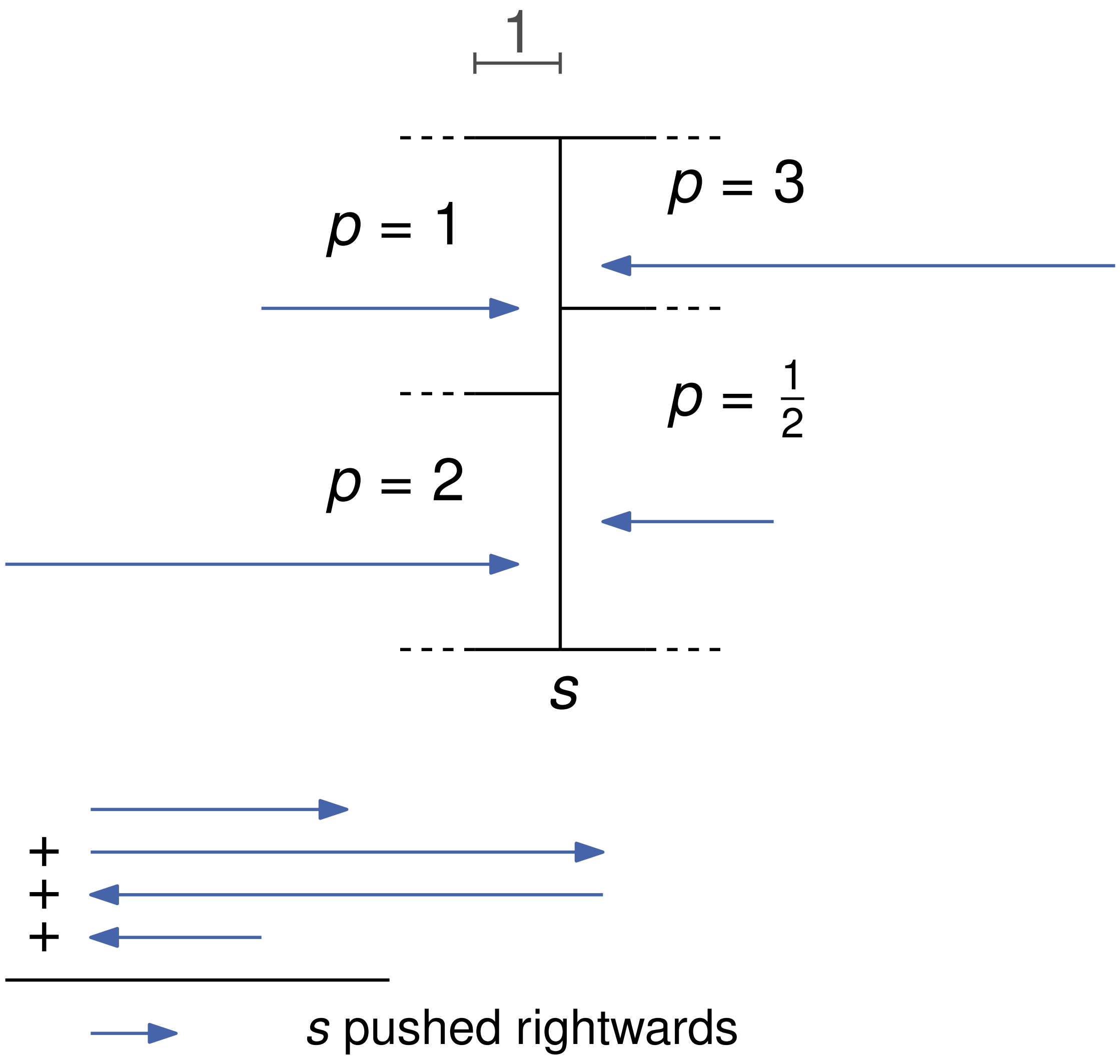
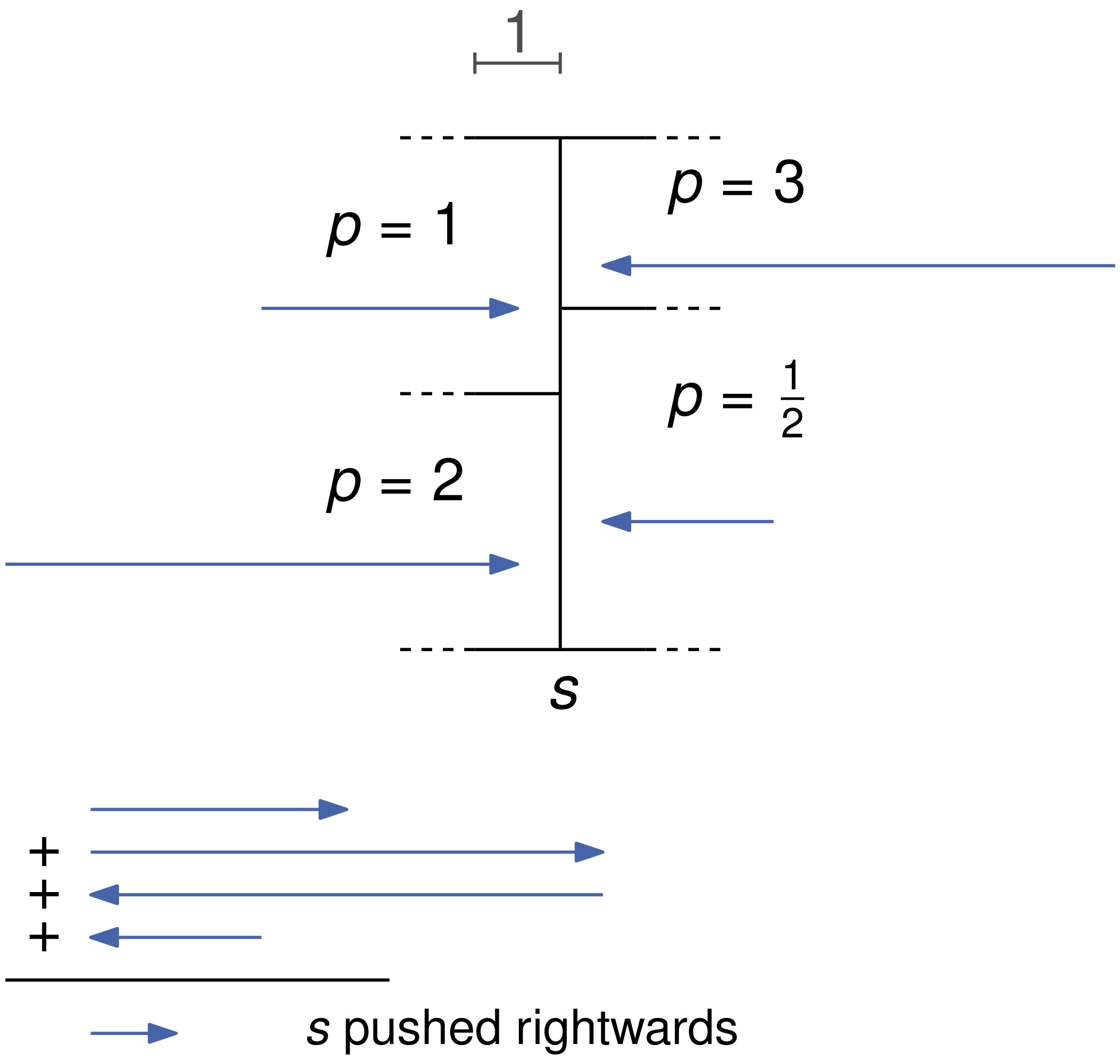


Algorithm

while “some” segment s is unbalanced

  push s in direction of force until it is balanced

Example



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while “some” segment *s* is unbalanced

push *s* in direction of force until it is balanced

History:

1998 Izumi, Takhashi and Kajitani implemented air-pressure method. Observed fast convergence.

2013 Convergence proved by Felsner.

Main Lemmas

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Lemma:
If all rectangles are in balance, then all segments are in balance.

Proof:
W.l.o.g. for a vertical segement s:

$$\mathcal{F}(s) = \sum_{r \text{ left of } s} p(r) \cdot \ell(r, s) - \sum_{r \text{ right of } s} p(r) \cdot \ell(r, s) = 0 \qquad \text{(since } p(r) = 1 \text{ for all } r\text{)}$$

$\square$

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Lemma:
If all segments are in balance, then all rectangles are in balance.

Proof more technical. And not (!) obvious.

Entropy

Definition (**Entropy**):

$$E(r) = -w(r) \cdot \log p(r) \qquad E(F) = \sum_r E(r) = \sum_r -w(r) \cdot \log p(r)$$

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Proof:

- Use that for $x \in \mathbb{R}_{>0}$ it holds that $\log(x) \geq (1 - \frac{1}{x})$. Equality if and only if $x = 1$.
- Thus: $\log p(r) \geq (1 - \frac{1}{p(r)}) = (1 - \frac{\text{area}(r)}{w(r)})$
- Thus: $E(r) = -w(r) \cdot \log p(r) \leq -w(r)(1 - \frac{\text{area}(r)}{w(r)}) = \text{area}(r) - w(r)$
- $E = \sum_r -w(r) \cdot \log p(r) \leq \sum_r \text{area}(r) - \sum_r w(r) = 1 - 1 = 0$ $\square$

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Shifting an unbalanced segment s into its balanced position increases the entropy.

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Proof idea:

- Let $E(t)$ be the entropy after shifting s by t.
- Force acting on s at time t is exactly $\frac{d}{dt}E(t)$.
- Force is positve in direction of shift.
  $\Rightarrow$ Change of entropy is positive.
  $\Rightarrow$ Entropy increases. $\square$

Putting it all together

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Theorem:
Let $F_0, F_1, F_2, \ldots$ be a sequence of floorplans where F_{i+1} is obtained from F_i by balancing an unbalanced segment.
If the selection of segments is “non-ignoring”, then the sequence converges to a floorplan F with $E(F) = 0$.

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Proof sketch:

- We know $E(F_i) \leq 0$ and $E(F_{i+1}) > E(F_i)$.
  $\Rightarrow$ Sequence converges.
- Assume (for contradiction) that $E(F) = a < 0$:
  - There is an unbalanced rectangle and thus an unbalanced segment s.
  - Balancing s gives increase Δ in entropy.
  - For all sufficiently large i balancing s in F_i increases entropy by at least $\Delta/2$. ↯
  - But: For all sufficiently large i we also have $E(F_i) > a - \Delta/2$. $\square$

Area-Universality of Plane Graphs

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Definition:
A plane graph G is **area-universal** if for every area assignment there is a straight-line drawing of G with the desired face areas.

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- stacked triangulations

  Biedl, Veláquez 2013
- 3-regular graphs

  Thomassen 1992
- 1-subdivisions of planar graphs

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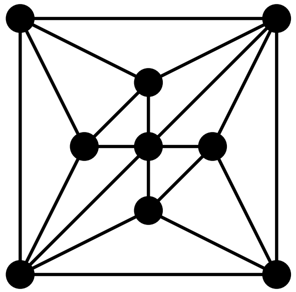
Not Area-Universal

- Eulerian triangulations
  Kleist 2018
- icosahedron, “butterfly graph”, ...
  Kleist 2018



Air-Pressure

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Pressure and **entropy** are defined as for floorplans:

$$p(f) = \frac{w(f)}{\text{area}(f)}$$

(f is an inner face, D a drawing)

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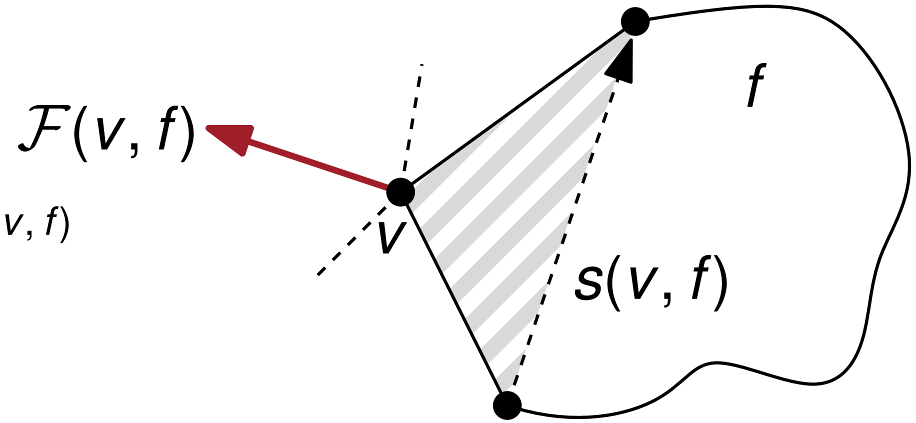
Forces:

$$\mathcal{F}(v, f) = p(f) \cdot \left|\left|s(v, f)\right|\right| \cdot n_{s(v,f)}$$

unit normal to $s(v, f)$

$$\mathcal{F}(v) = \sum_{f|(v \text{ on } f)} \mathcal{F}(v, f)$$



Shifting a Vertex

Lemma: (Kleist 2018)
In a triangulation an unbalanced vertex can always be shifted into a balanced position.

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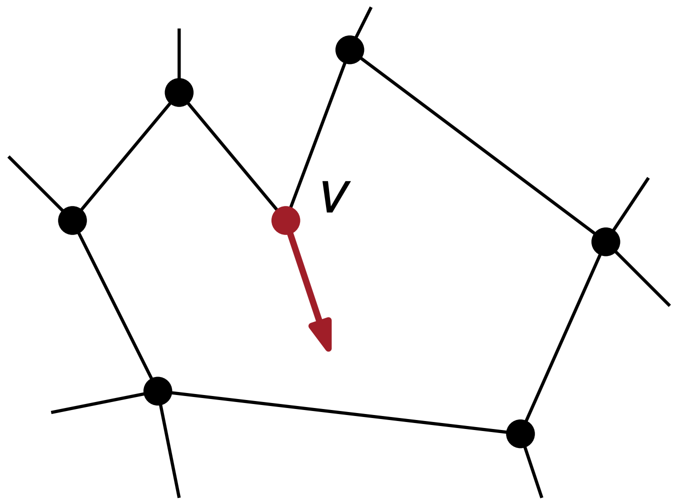
General plane graphs:



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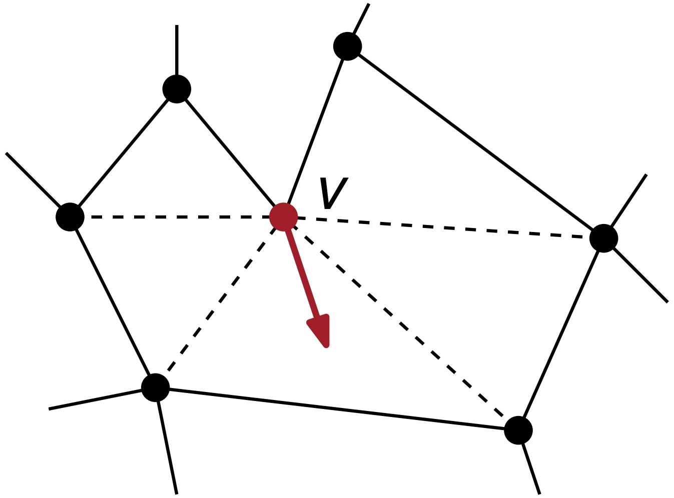

1) triangulate around *v*

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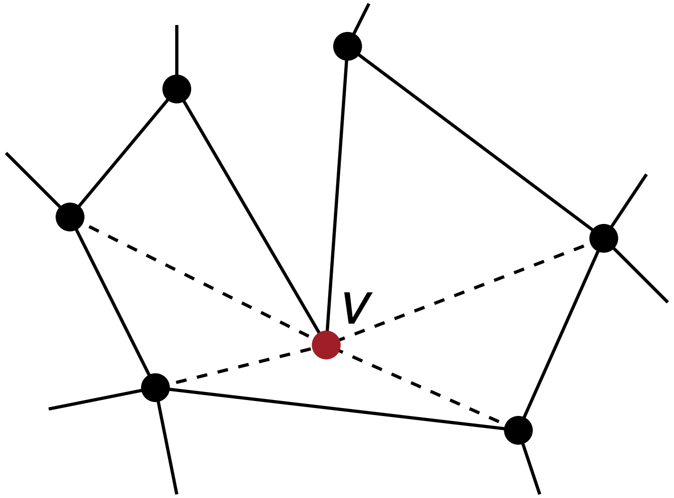

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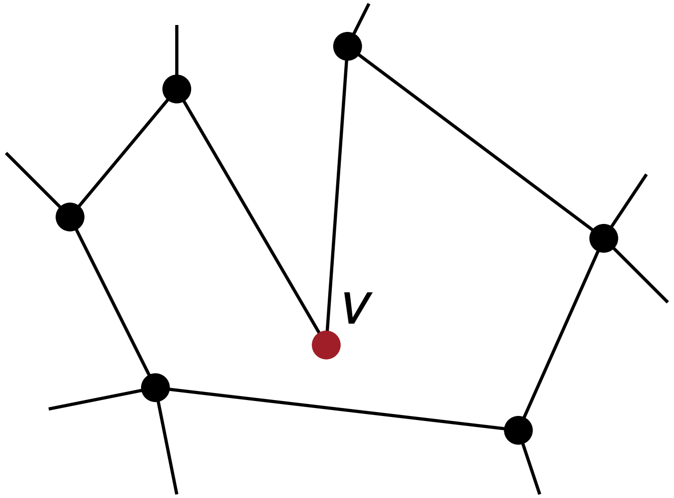

1) triangulate around v
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3) remove dashed edges $\rightsquigarrow$ "pressure equalization"

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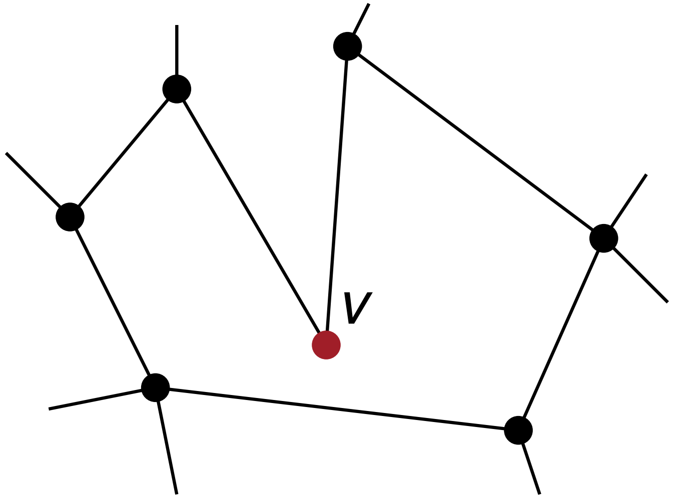

1) triangulate around v
2) shift v until in balance
3) remove dashed edges $\rightsquigarrow$ “pressure equalization”

Steps 2) and 3) both increase entropy.

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Deadlocks

Lemma: If all faces are in balance, then all vertices are in balance.

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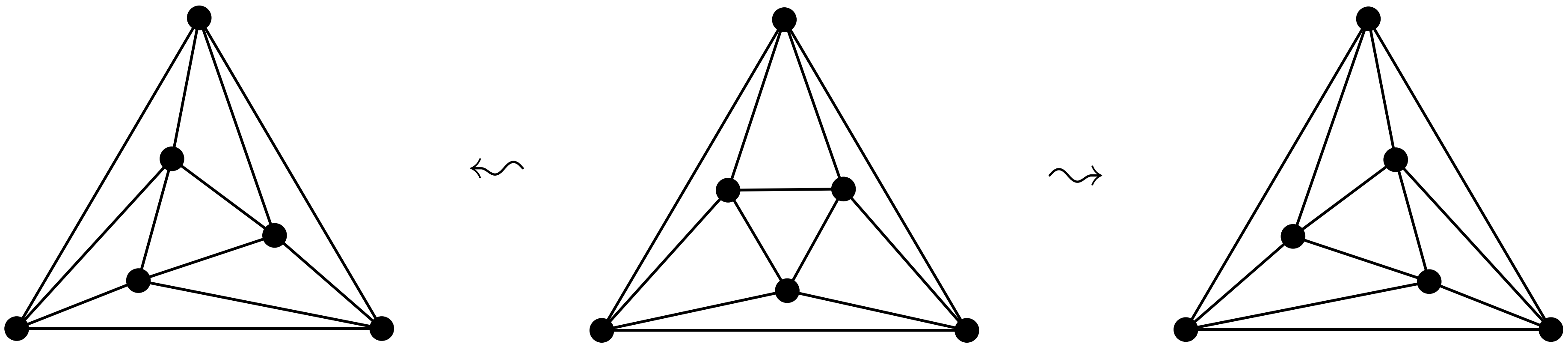
Deadlocks

Lemma: If all faces are in balance, then all vertices are in balance.

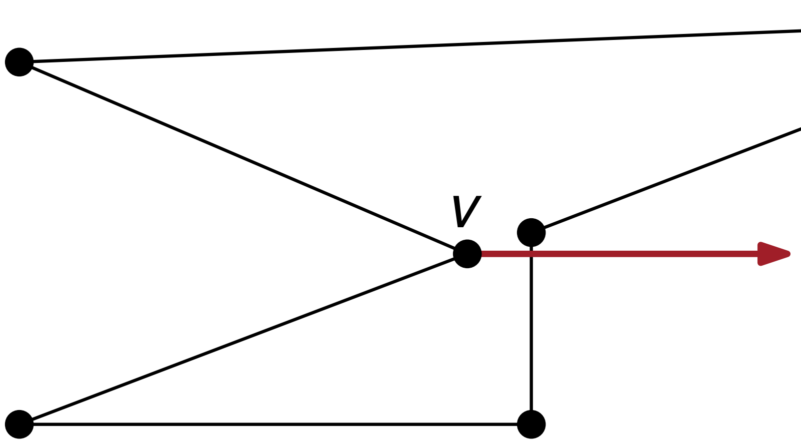
But: The converse is not true!



equiareal | **Deadlock** all forces are 0 | equiareal

Planarity

Even without deadlocks, a vertex might not be moveable:



Vertex v will move towards the edge, but never enter the pocket.

Deadlock-Free Graph Classes

Theorem: (Kleist 2018)
There are no deadlocks for 3-regular graphs.

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Possible way to reprove Thomassen's theorem that 3-regular graphs are area-universal.

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Lower the *subdivision number* of triangulations and quadrangulations.

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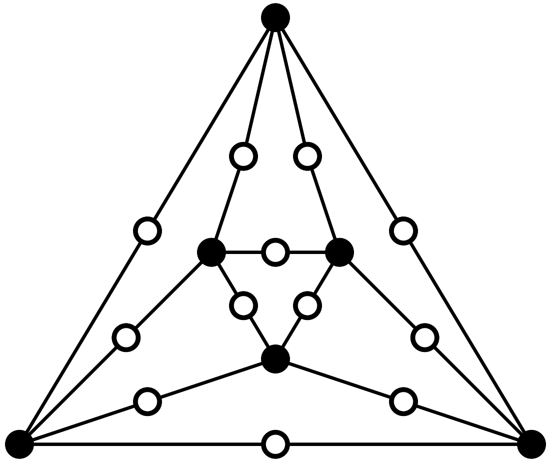
Subdivision Number

Theorem: (Kleist 2018)
1-subdivisions of planar graphs are area-universal.

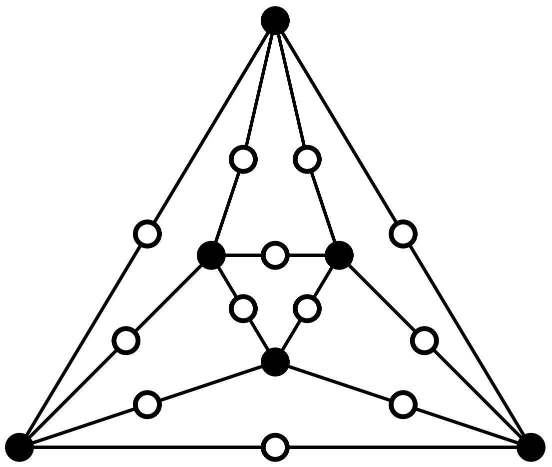


Subdivision Number

Theorem: (Kleist 2018)
1-subdivisions of planar graphs are area-universal.

Question: How many subdivisions are necessary to obtain an area-universal graph?

$\rightsquigarrow$ subdivision number $s(n)$



Triangulations: $s(n) \leq 3n - 6$

Quadrangulations: $s(n) \leq n - 2$

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What if 2-degenerate graphs were area-universal?

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$\rightsquigarrow s(n) \leq n - 3 \quad (3n - 6)$ for n-vertex triangulations

$\rightsquigarrow s(n) \leq \left\lfloor \frac{2n}{3} \right\rfloor \quad (n - 2)$ for n-vertex quadrangulations

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New question:
How many edges need to be subdivided (or deleted) to obtain a 2-degenerate graph?

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Triangulations: exactly $n - 3$

Quadrangulations: $\leq \frac{n}{3}$

$\geq \frac{n}{4}$

Triangulations

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Theorem:
For every n-vertex triangulation there exists a 2-degenerate subdivision with exactly $n-3$ subdivision vertices.

Triangulations

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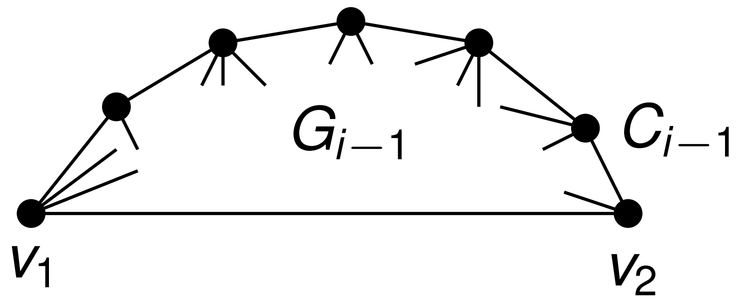
For every n-vertex triangulation there exists a 2-degenerate subdivision with exactly $n-3$ subdivision vertices.

Proof:

- Consider a canonical ordering $v_1, \ldots, v_n$. For every $4 \leq i \leq n$:
  - $G_{i-1} = G[v_1, \ldots, v_{i-1}]$ is a 2-connected inner triangulation
  - with outer cycle C_{i-1} containing egde $v_1 v_2$
  - and v_i is outside of C_{i-1}
  - with at least two consecutive neighbors along C_{i-1}.
- In each step, subdivide all “inner” edges.
- Subdivides all but $(n-2) \cdot 2 + 1 = 2n - 3$ edges. □



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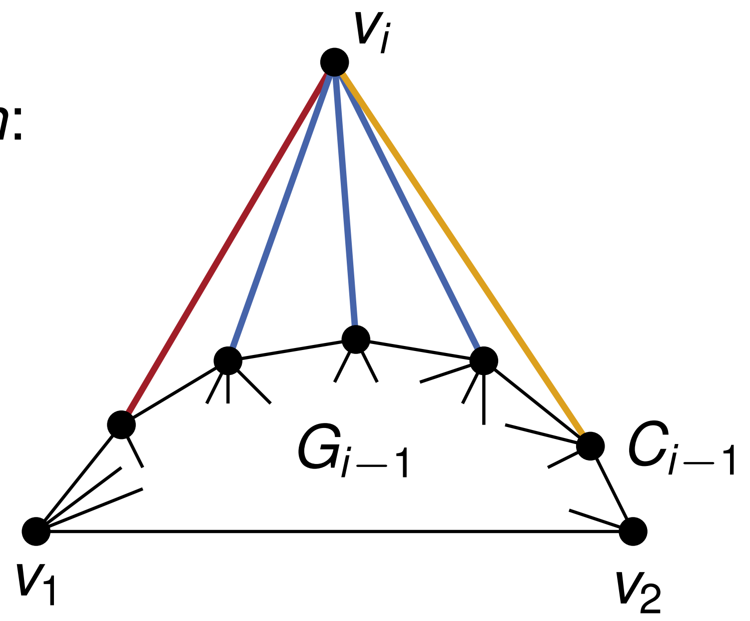


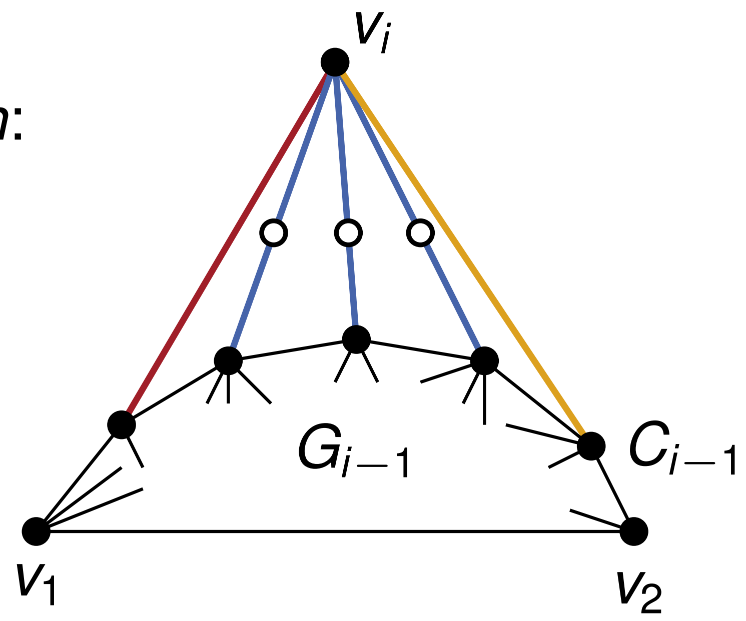
□

Triangulations

Theorem:
For every n-vertex triangulation there exists a 2-degenerate subdivision with exactly $n-3$ subdivision vertices.

Proof:

- Consider a canonical ordering $v_1, \dots, v_n$. For every $4 \leq i \leq n$:
  - $G_{i-1} = G[v_1, \dots, v_{i-1}]$ is a 2-connected inner triangulation
  - with outer cycle C_{i-1} containing egde $v_1 v_2$
  - and v_i is outside of C_{i-1}
  - with at least two consecutive neighbors along C_{i-1}.
- In each step, subdivide all “inner” edges.
- Subdivides all but $(n-2) \cdot 2 + 1 = 2n - 3$ edges.



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Questions?