Primal-Dual Cops and Robber

EuroCG 2023 · 29.3.2023

Minh Tuan Ha, Paul Jungeblut, Torsten Ueckerdt
Cops & Robber

2-Players: \( k \) Cops, 1 Robber

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Cops & Robber

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- Cops go first.
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**Cop number** \( c(G) \):
How many cops are necessary to capture the robber?
Primal-Dual Cops & Robber

- Played in a plane graph $G$. 
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- Played in a **plane** graph $G$.
- Cops play in the **dual** graph $G^*$:
  - $\sim$ **Face-Cops**
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- Robber plays in the primal graph $G$. (as in the classical game)
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- Cops win by occupying all faces incident to the robber.
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- Cops win by occupying all faces incident to the robber.

**Primal-dual cop number $c^*(G)$:**
How many face-cops are necessary to capture the robber?
Results

**Trivial lower bound:**
\[ c^*(G) \geq \Delta(G) \quad \text{(if } G \text{ is 2-connected)} \]

\[ \uparrow \quad \text{max. degree} \]
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**max. degree**

**Question:**
Is there an upper bound on \( c^*(G) \) in terms of \( \Delta(G) \)?
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\[ \text{max. degree} \]

Question:
Is there an upper bound on \( c^*(G) \) in terms of \( \Delta(G) \)?

Theorem:
For a plane graph \( G \):
\[ c^*(G) \leq 3 \quad \text{if} \quad \Delta(G) \leq 3 \]
\[ c^*(G) \leq 12 \quad \text{if} \quad \Delta(G) \leq 4 \]
\[ c^*(G) \in \Omega(\sqrt{\log(n)}) \quad \text{if} \quad \Delta(G) \geq 5 \]
$c^*(G) \leq 3 \quad \text{for} \quad \Delta(G) \leq 3$

(Simplification: 3-regular, 2-connected)
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$c^*(G) \leq 3$ for $\Delta(G) \leq 3$
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- Cops *choose* their target faces.

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- Cops choose their target faces.
- Cops update their target faces.
  - $d_b$ and $d_r$ increased by 1
  - $d_g$ did not change
  - each cop may take one step
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\[ d_b + d_r + d_g \text{ decreases during each cop-turn} \]

(There is an edge case left in the endgame.)
\[ c^*(G) \leq 12 \quad \text{for} \quad \Delta(G) \leq 4 \]
(Simplification: 4-regular, 2-connected)

**Idea:** Four face-cops can simulate a vertex cop.
$$c^*(G) \leq 12 \text{ for } \Delta(G) \leq 4$$

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c∗(G) ≤ 12 for Δ(G) ≤ 4
(Simplification: 4-regular, 2-connected)

**Idea:** Four face-cops can simulate a vertex cop.

**Theorem:** (Aigner, Fromme 1984)
c(G) ≤ 3 for all planar graphs G.

4 · 3 = 12 face-cops always suffice
(in planar graphs with Δ(G) ≤ 4)
Primal-Dual Cops and Robber

Theorem: (Nisse, Suchan 2008)

\[ c_{p,q}(G_{n,n}) \in \Omega(\sqrt{\log(n)}) \] for the \( n \times n \)-grid graph \( G_{n,n} \).

\[ c^*(G) \text{ is unbounded for } \Delta(G) \geq 5 \]

\( G_{4,4} \)

variant of classical game

\( p < q \): cop and robber speeds (edges per turn)
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Idea: Simulate this in our primal-dual variant.

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**Theorem:** (Nisse, Suchan 2008)
\[ c_{p,q}(G_{n,n}) \in \Omega\left(\sqrt{\log(n)}\right) \] for the \( n \times n \)-grid graph \( G_{n,n} \).

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**Idea:** Simulate this in our primal-dual variant.

grid edge:

subdivide \( h \) times
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variant of *classical* game
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grid edge:
subdivide \( h \) times

\[ h + 1 \text{ steps} \]
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\(p < q\): cop and robber speeds (edges per turn)

Idea: Simulate this in our primal-dual variant.

grid edge: subdivide \(h\) times

\[h + 1\] steps

\(\approx \frac{3}{2}h\) steps
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$\Delta = 5$

$G_{4,4}$
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**Robber strategy:**
- copy strategy of Nisse and Suchan
  - robber moves between grid vertices “o”
  - “rounds” face-cop to nearest grid vertex
- inner rings $\leadsto$ no shortcuts for face-cops
- robber is faster $\leadsto c^*(G) \in \Omega(\sqrt{\log(n)})$
Open Problems

**Problem 1**

Find exact bounds:

- $\Delta(G) \leq 3$: $c^*(G) = 3$
- $\Delta(G) \leq 4$: $4 \leq c^*(G) \leq 12$
- $\Delta(G) \leq 5$: $c^*(G) \in \Omega(\sqrt{\log(n)})$
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**Problem 2**

Generalize the game:

- Consider graphs with crossing-free embeddings on other surfaces
- Use cycle double cover instead of faces