Primal-Dual Cops and Robber

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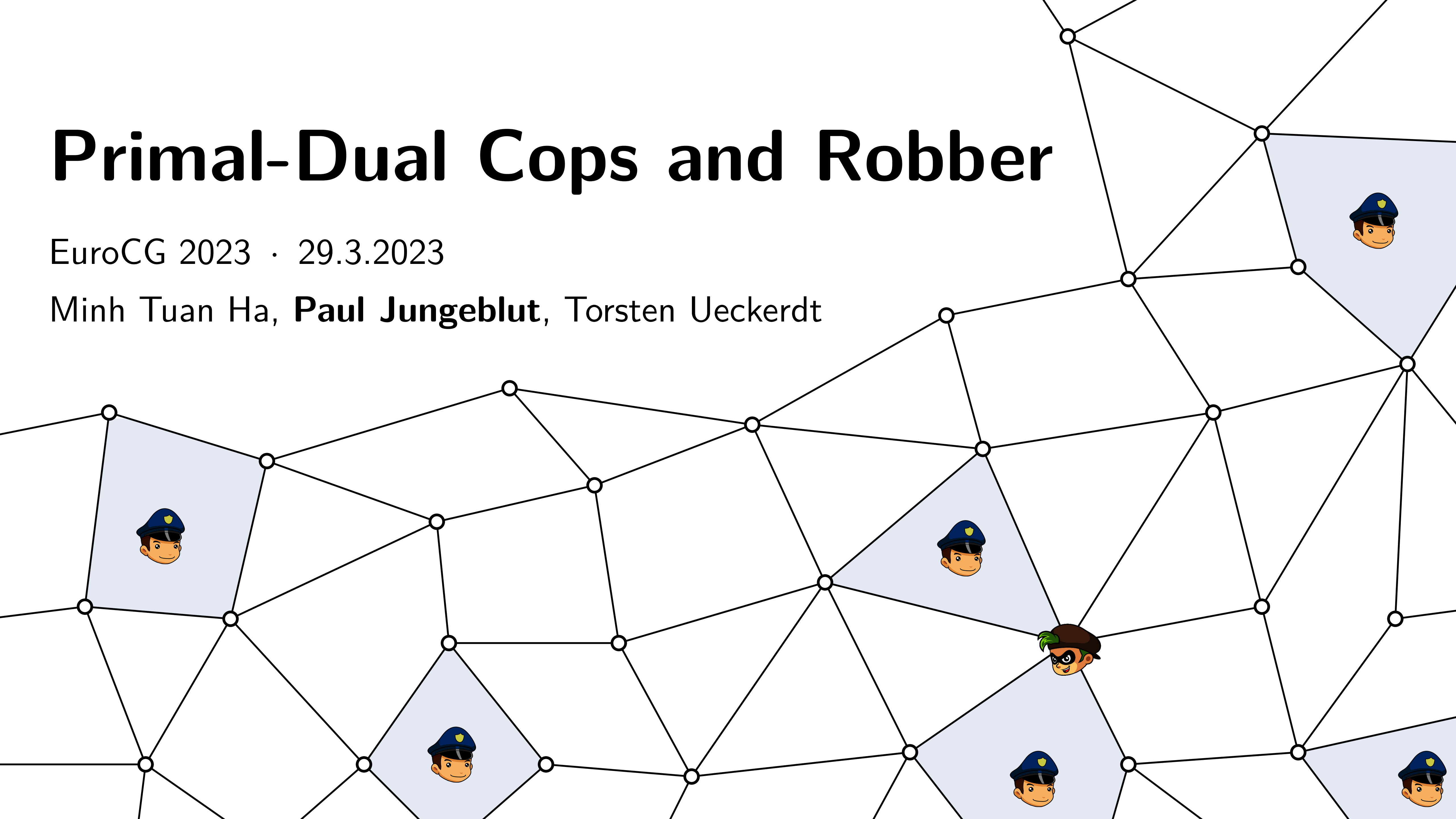
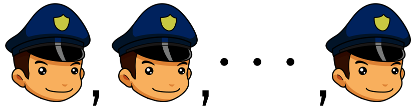
Minh Tuan Ha, **Paul Jungeblut**, Torsten Ueckerdt

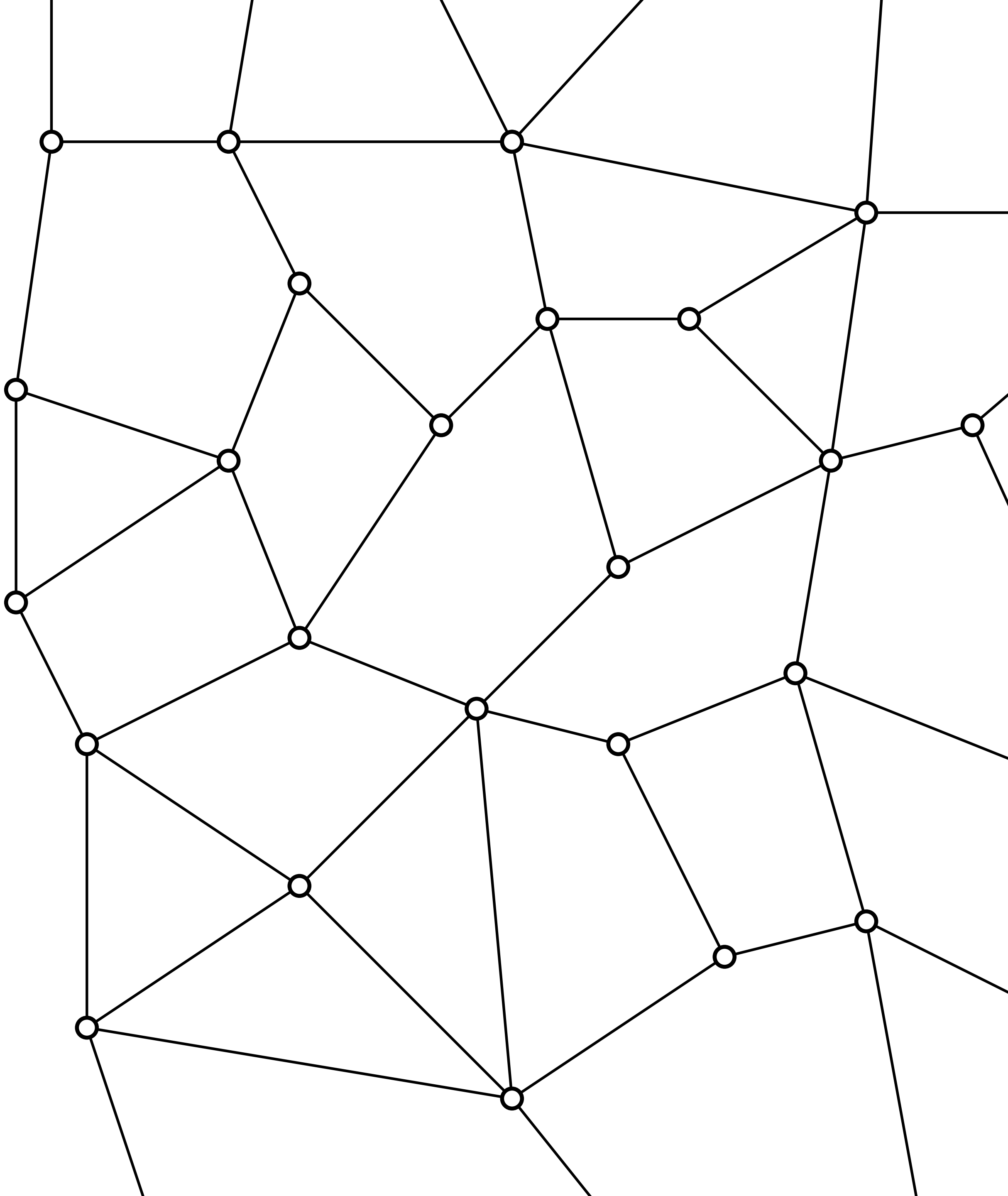


Cops & Robber

2-Players: k Cops 

1 Robber 

Rules:



Cops & Robber

2-Players: k Cops , , $\cdots$, 

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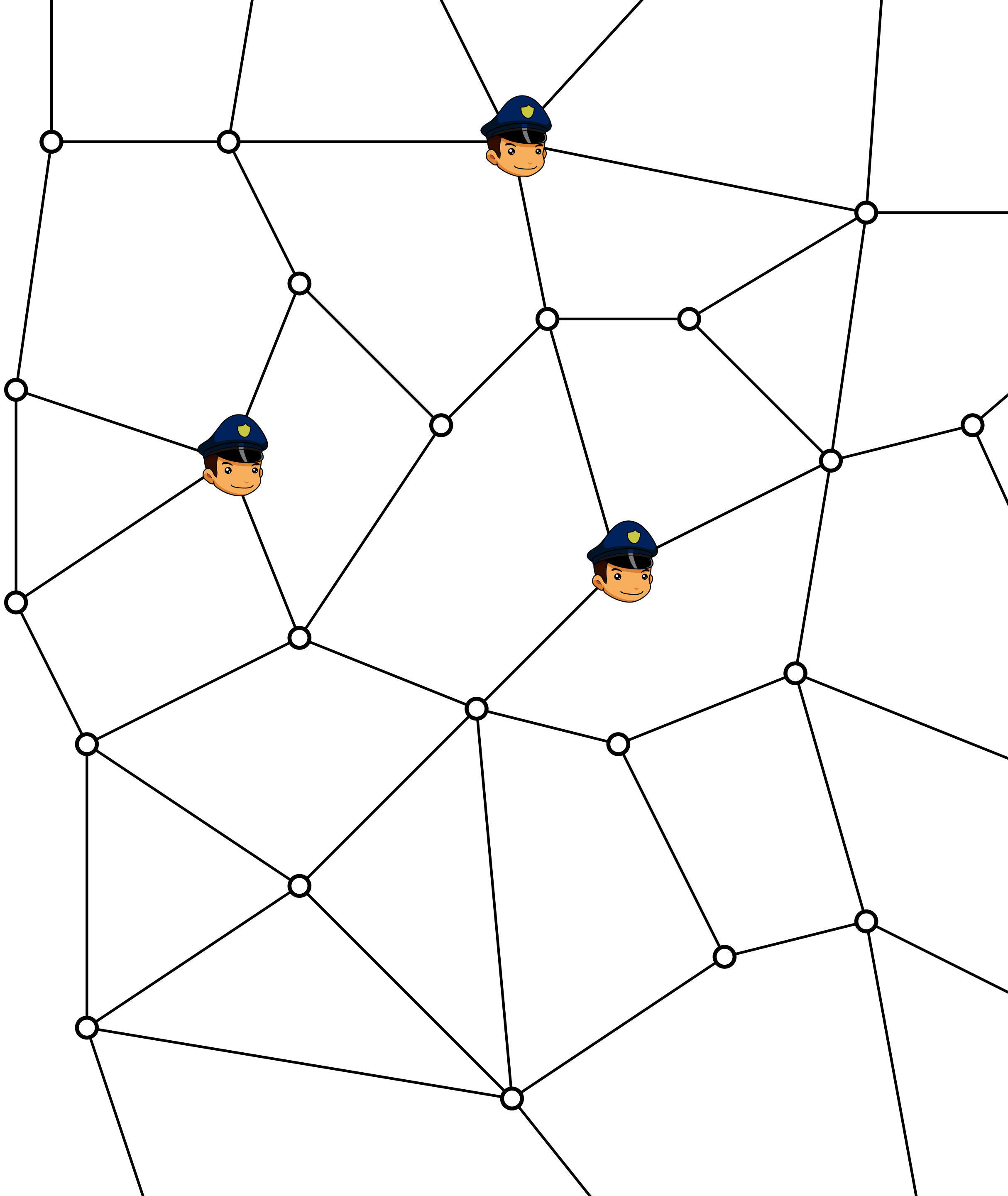

Cops & Robber

2-Players: k Cops

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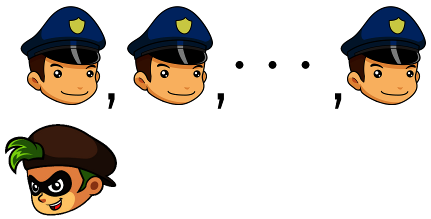

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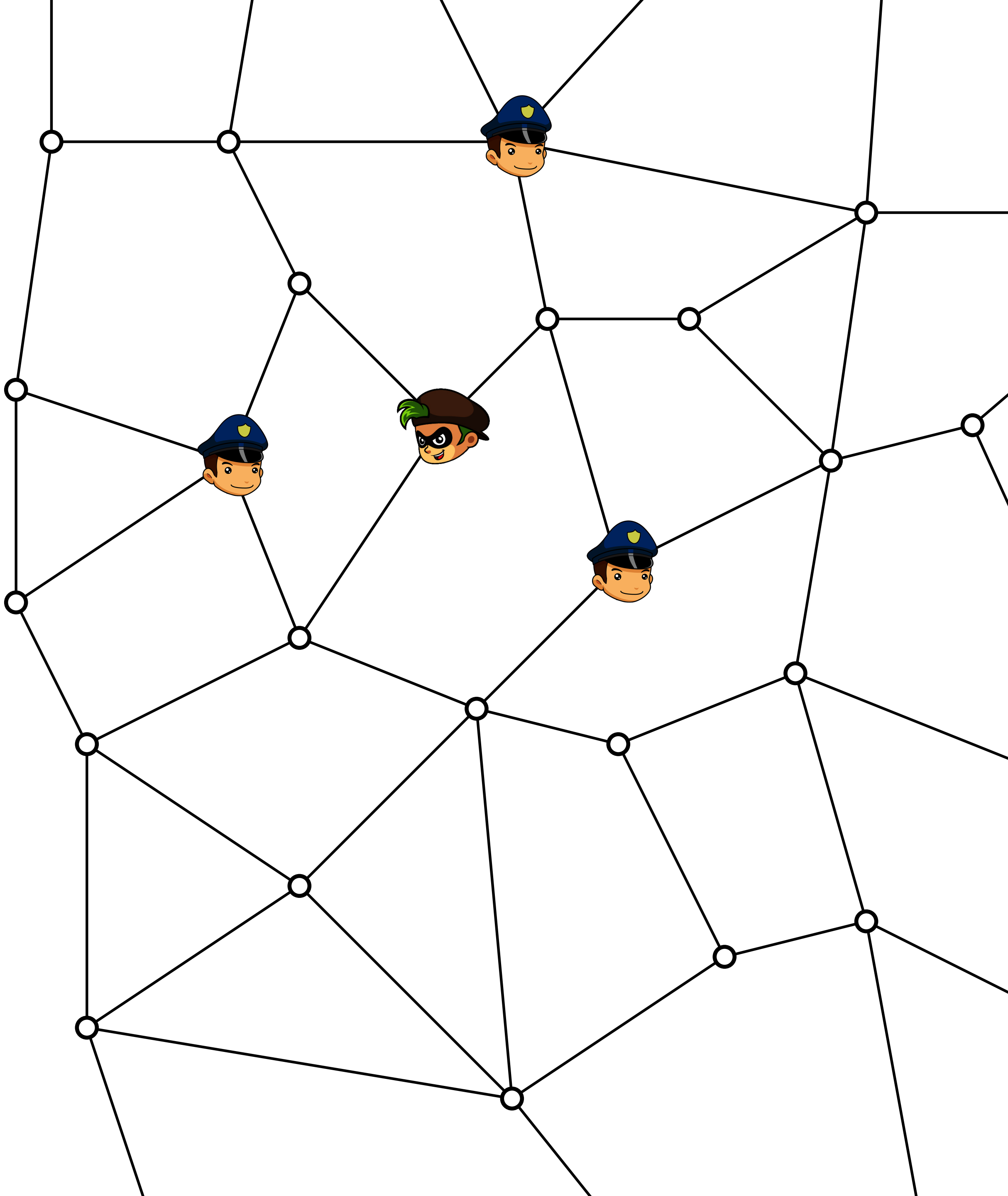

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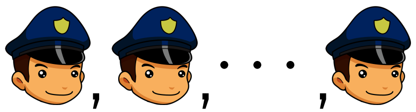
2-Players: k Cops 

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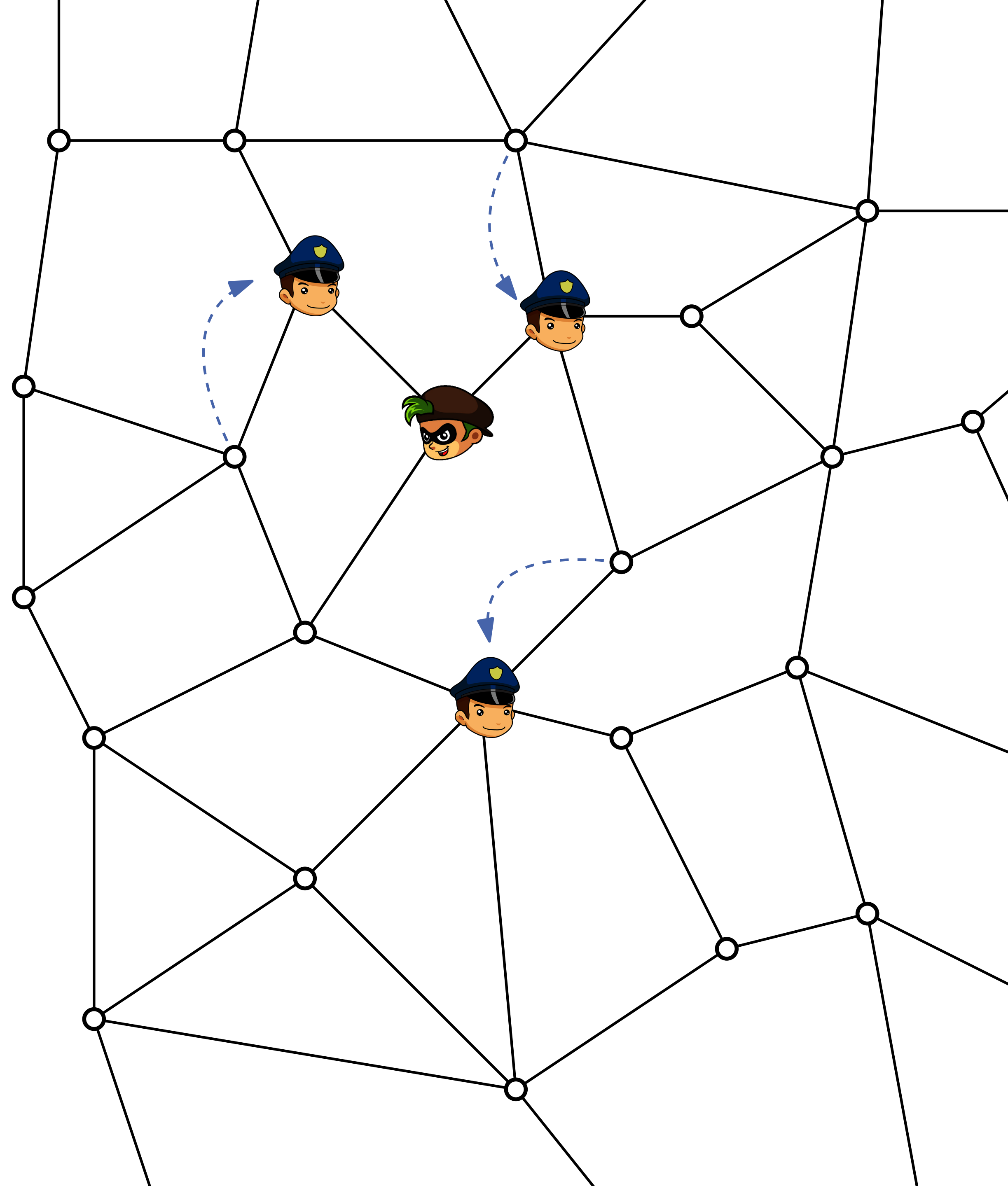
- Moves are between adjacent vertices.



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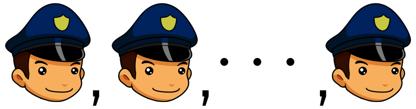
2-Players: k Cops 

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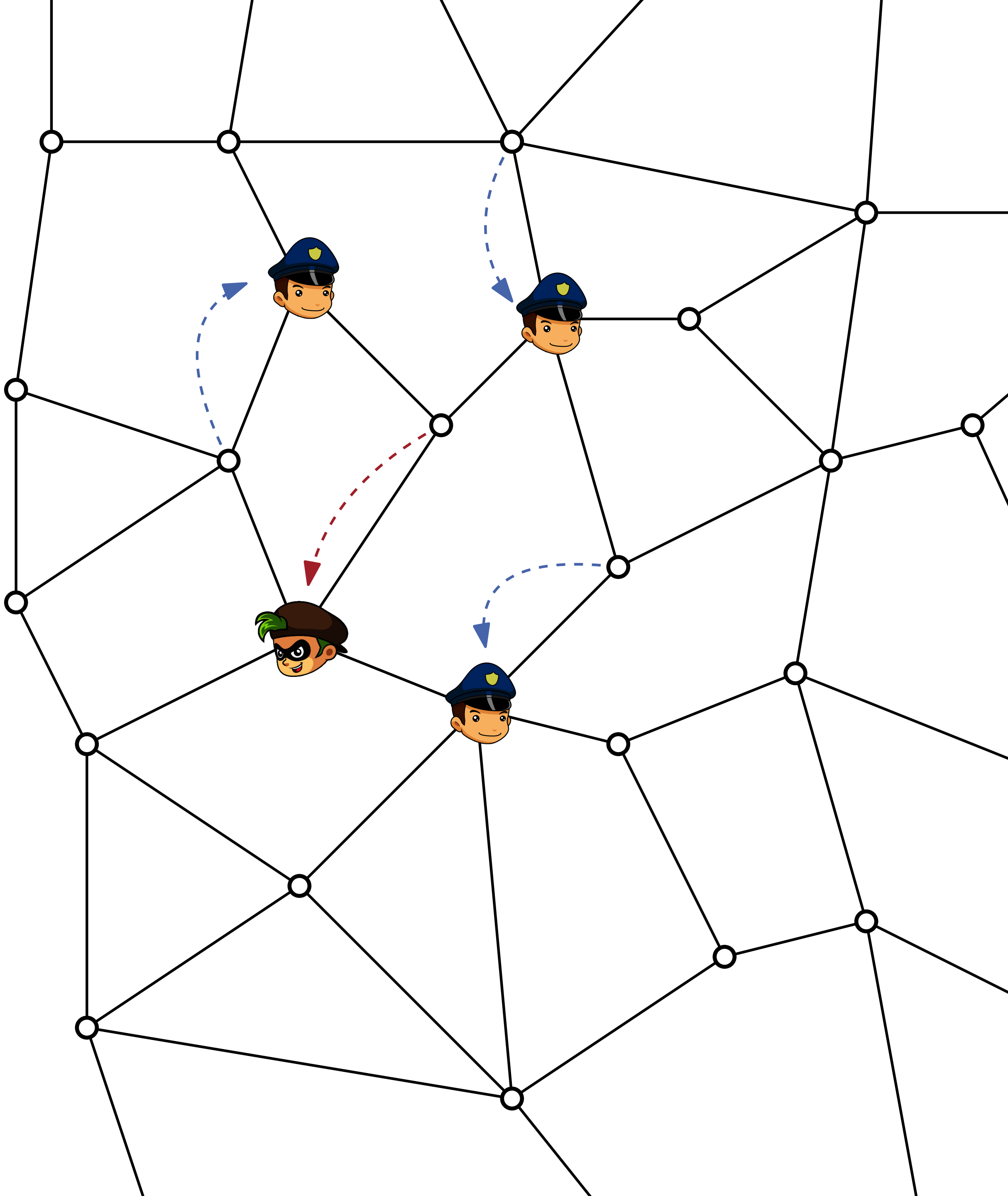

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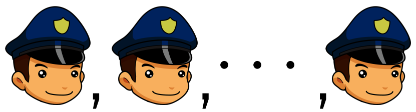
2-Players: k Cops 

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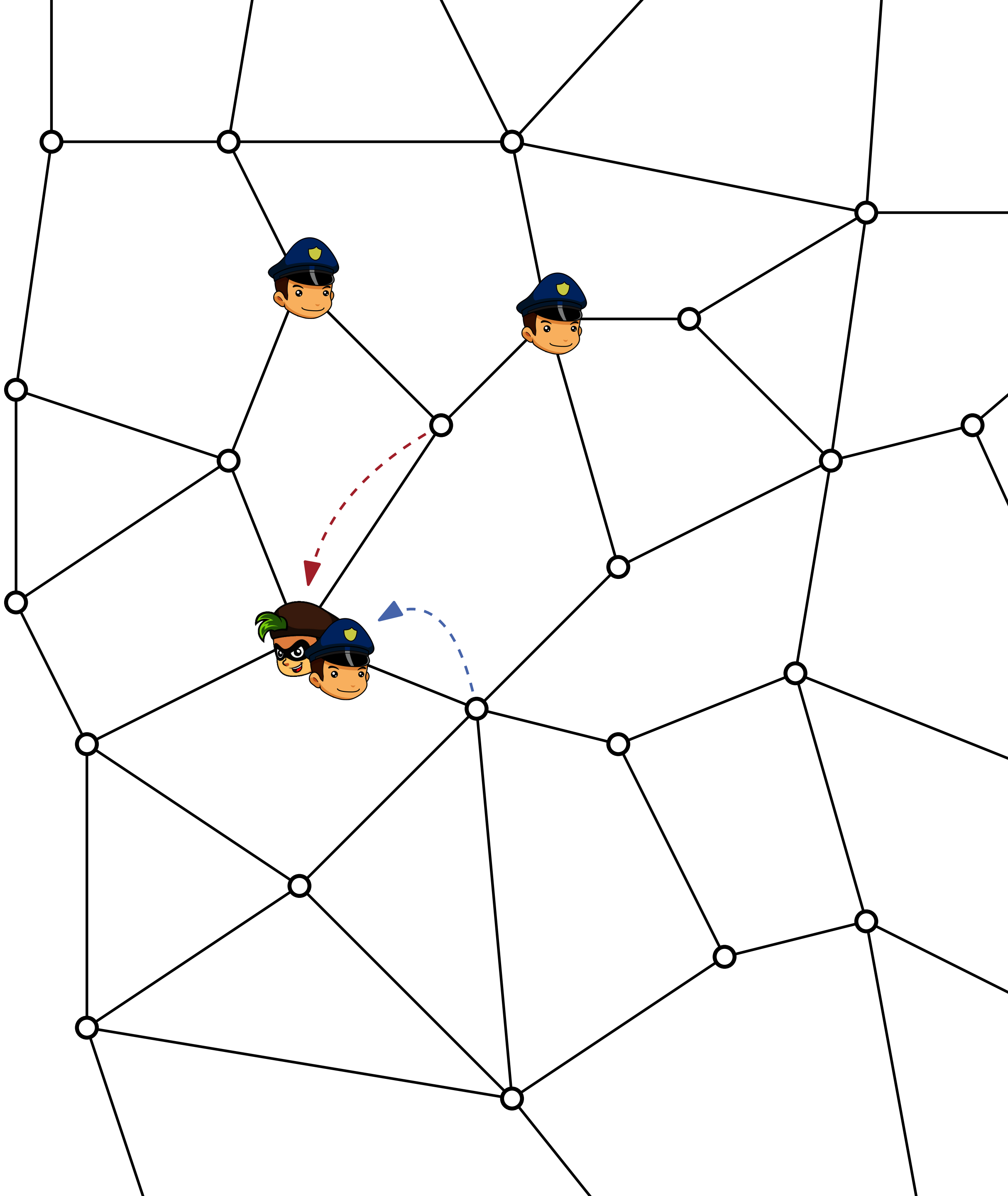
- Cops win by capturing the robber.



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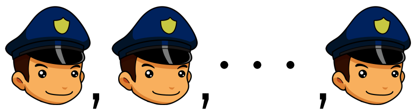
2-Players: k Cops 

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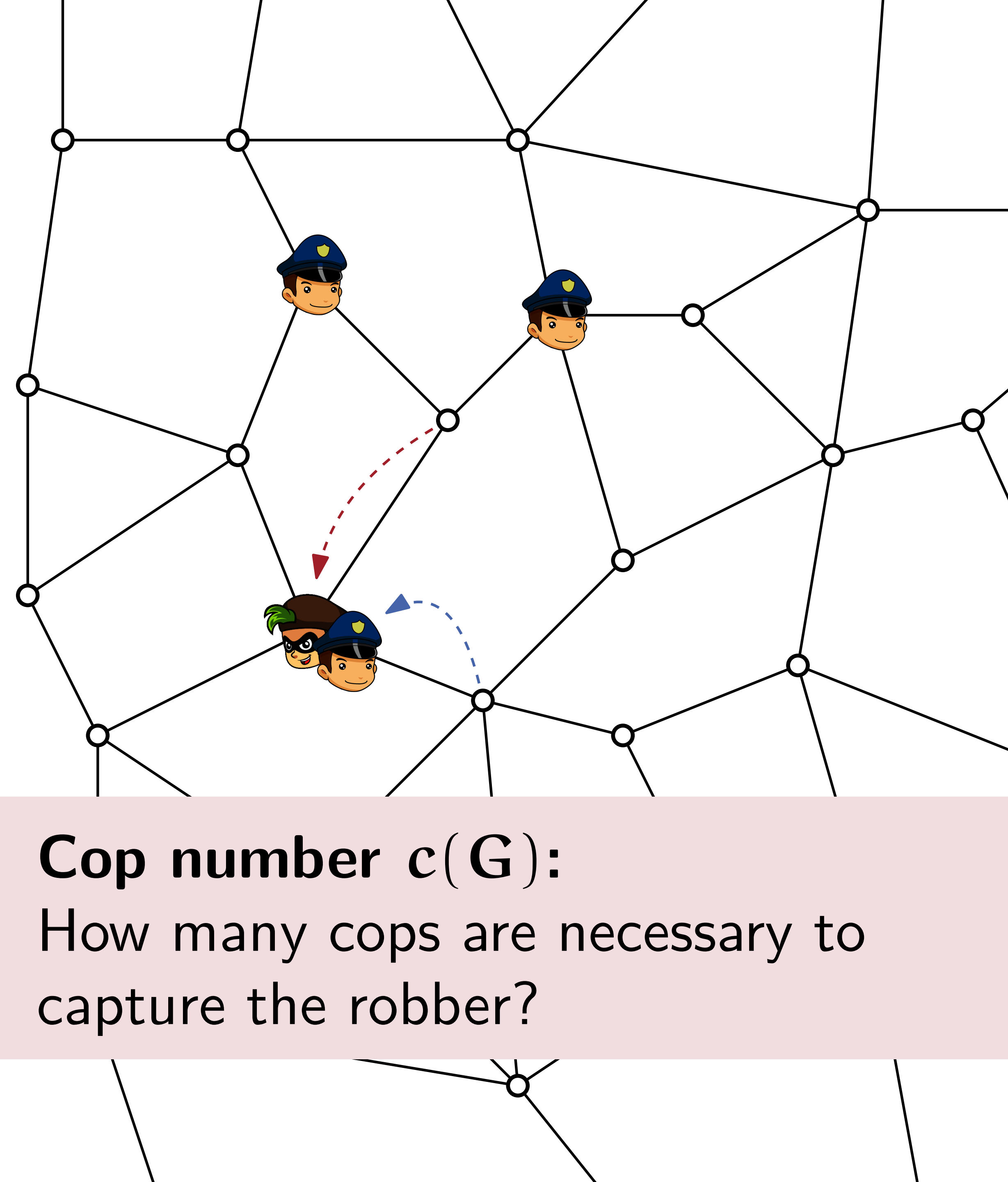
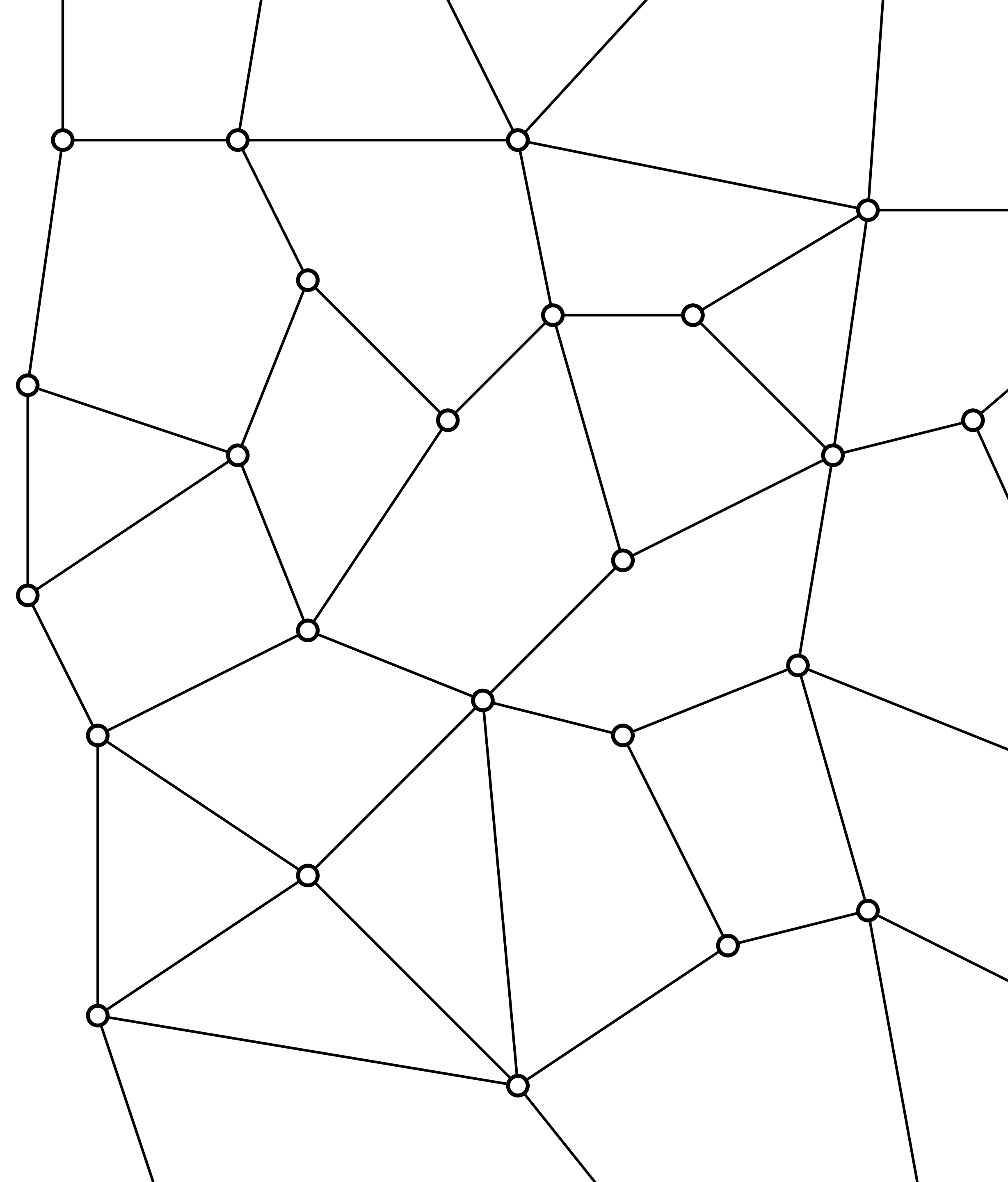


Cop number $c(G)$:
How many cops are necessary to capture the robber?

Primal-Dual Cops & Robber

- Played in a plane graph G.



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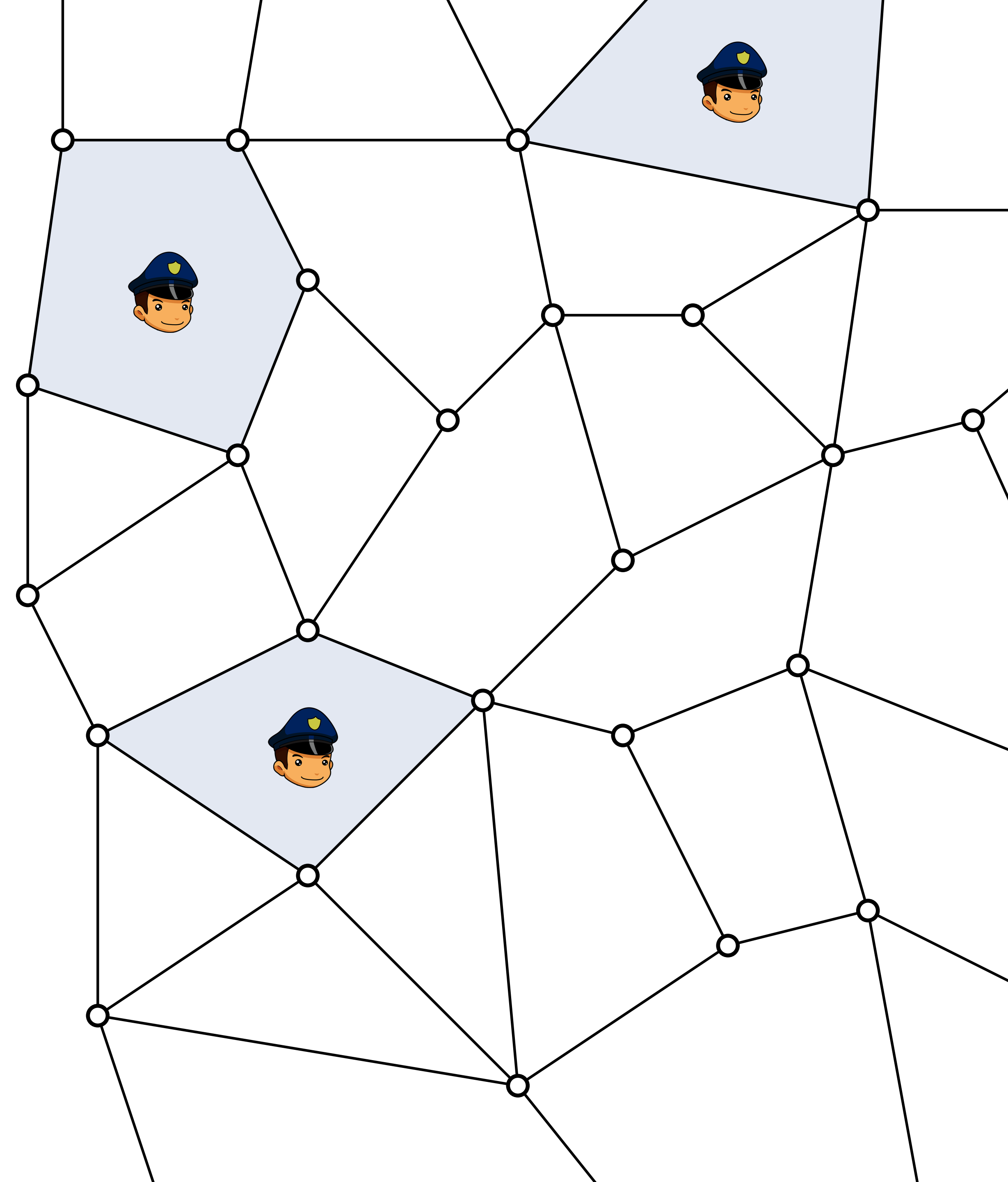
  ⤳ Face-Cops



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- Played in a plane graph G.
- Cops play in the dual graph G^*:
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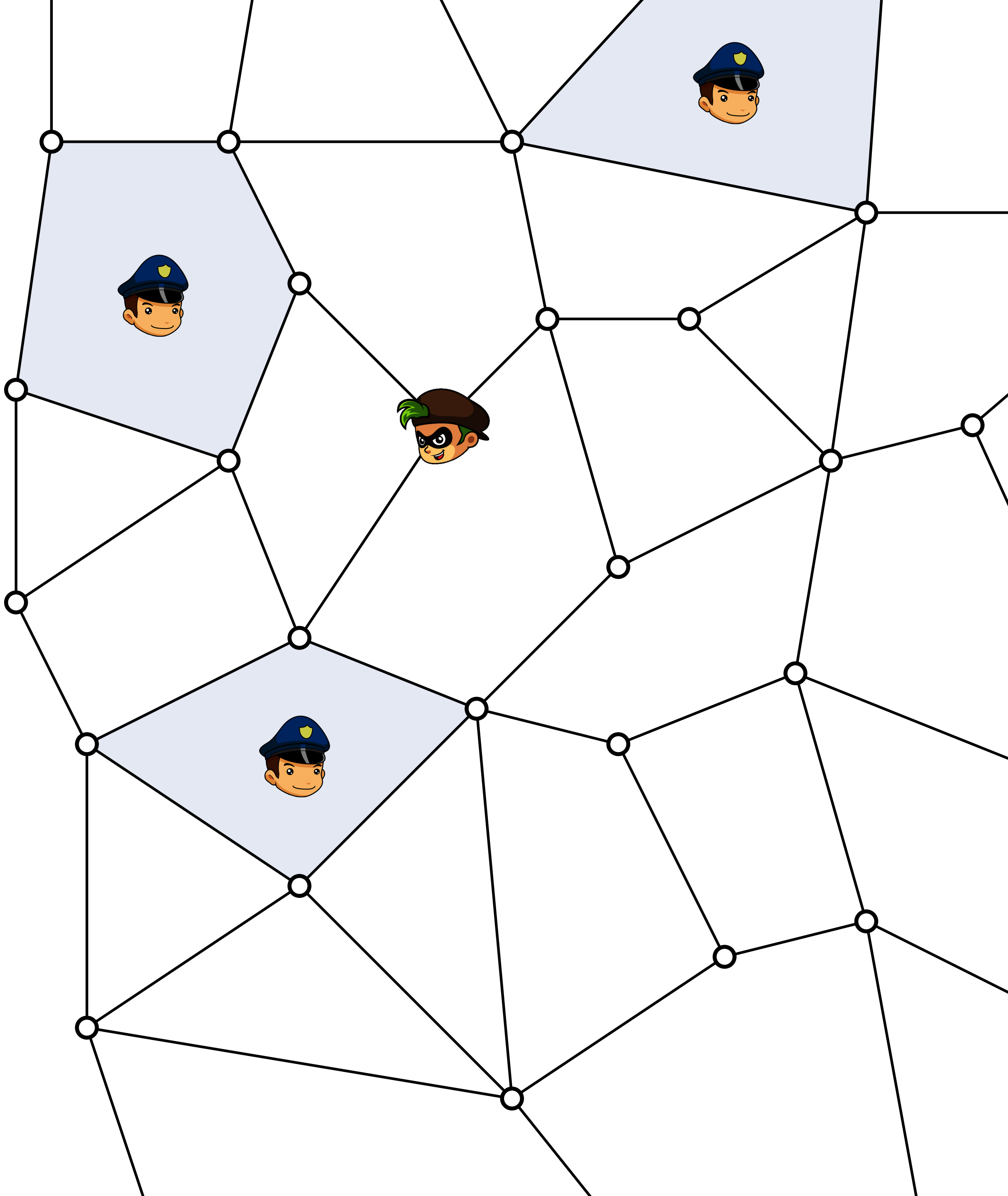
- Robber plays in the primal graph G.
  (as in the classical game)



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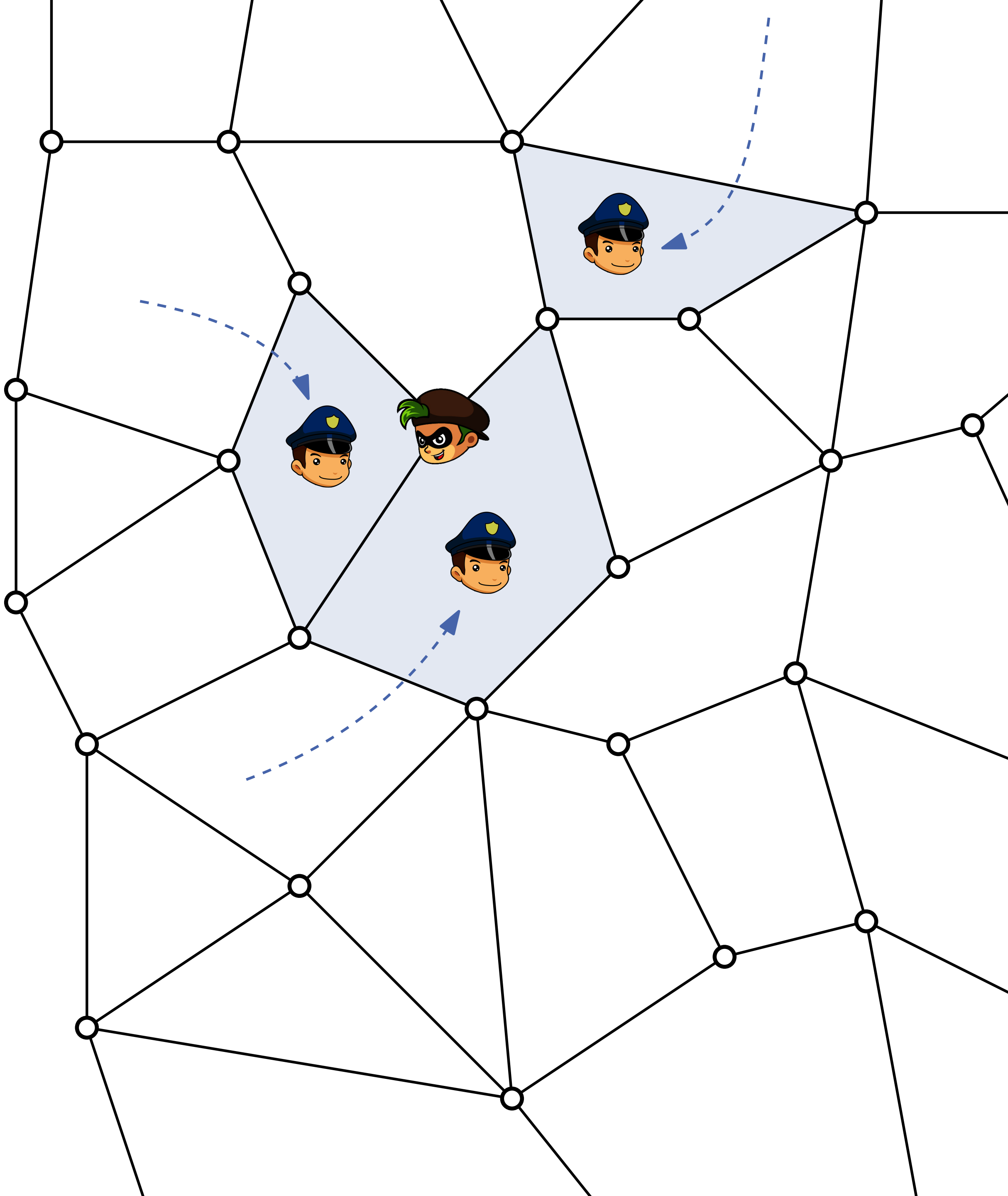

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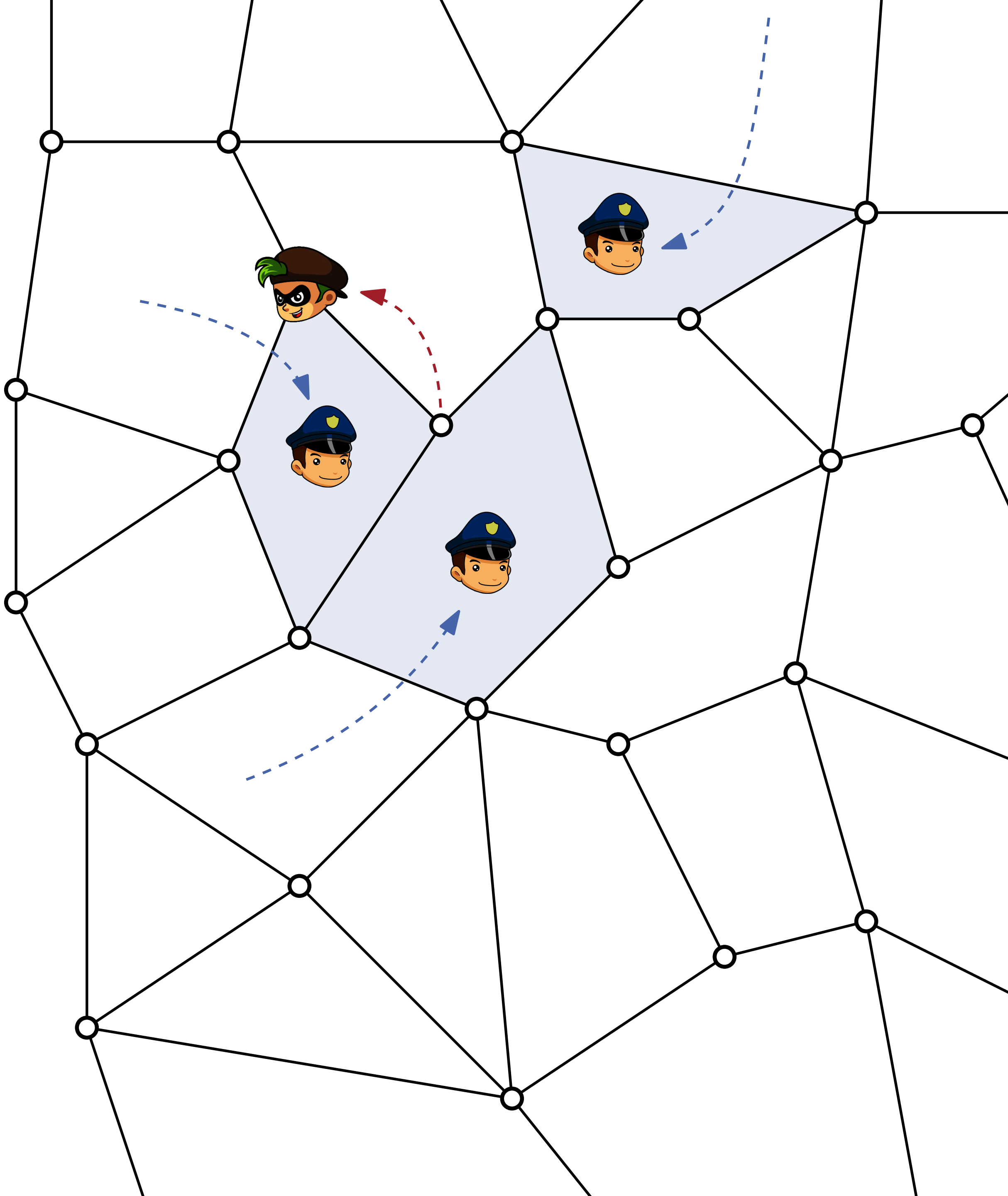

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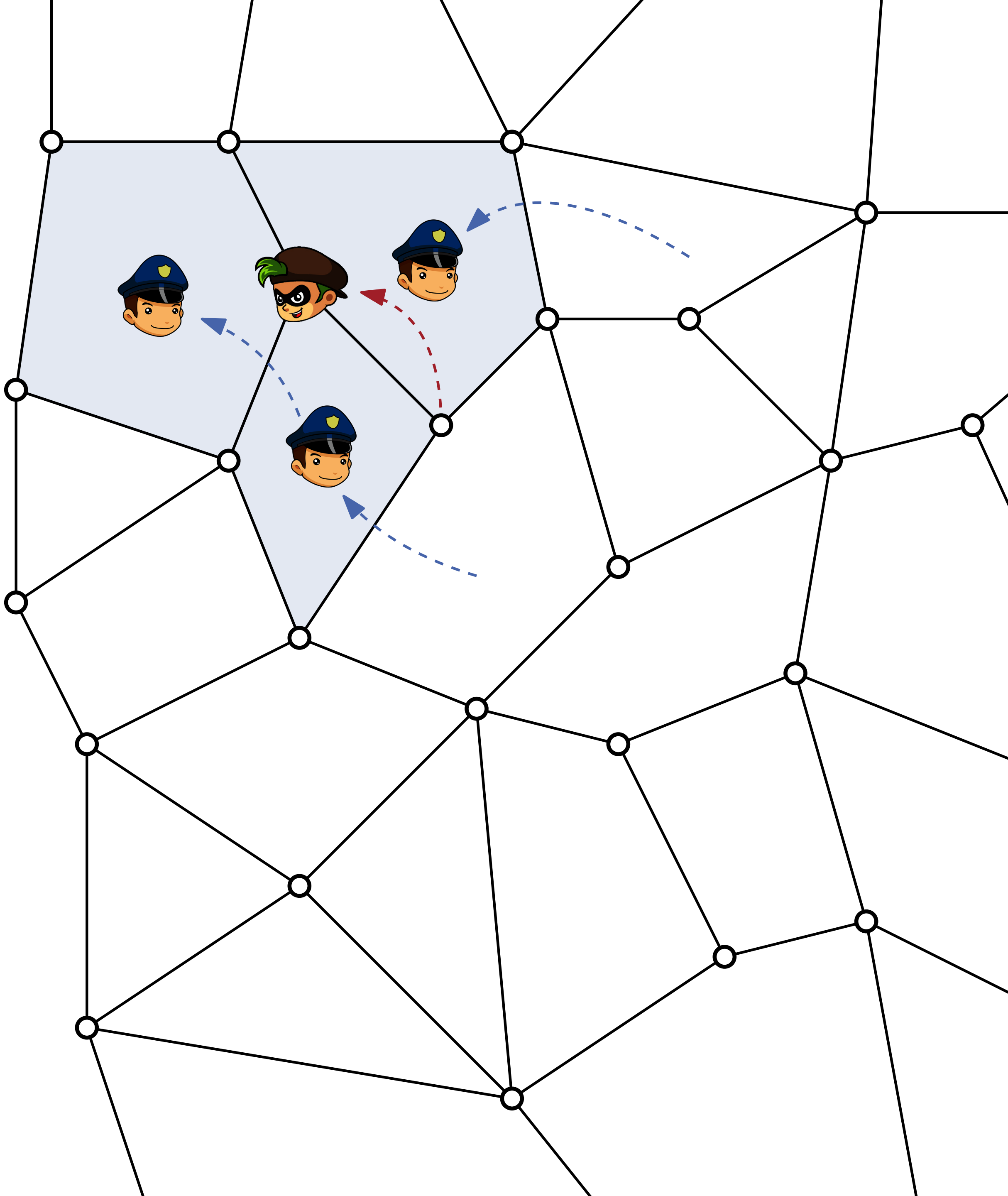

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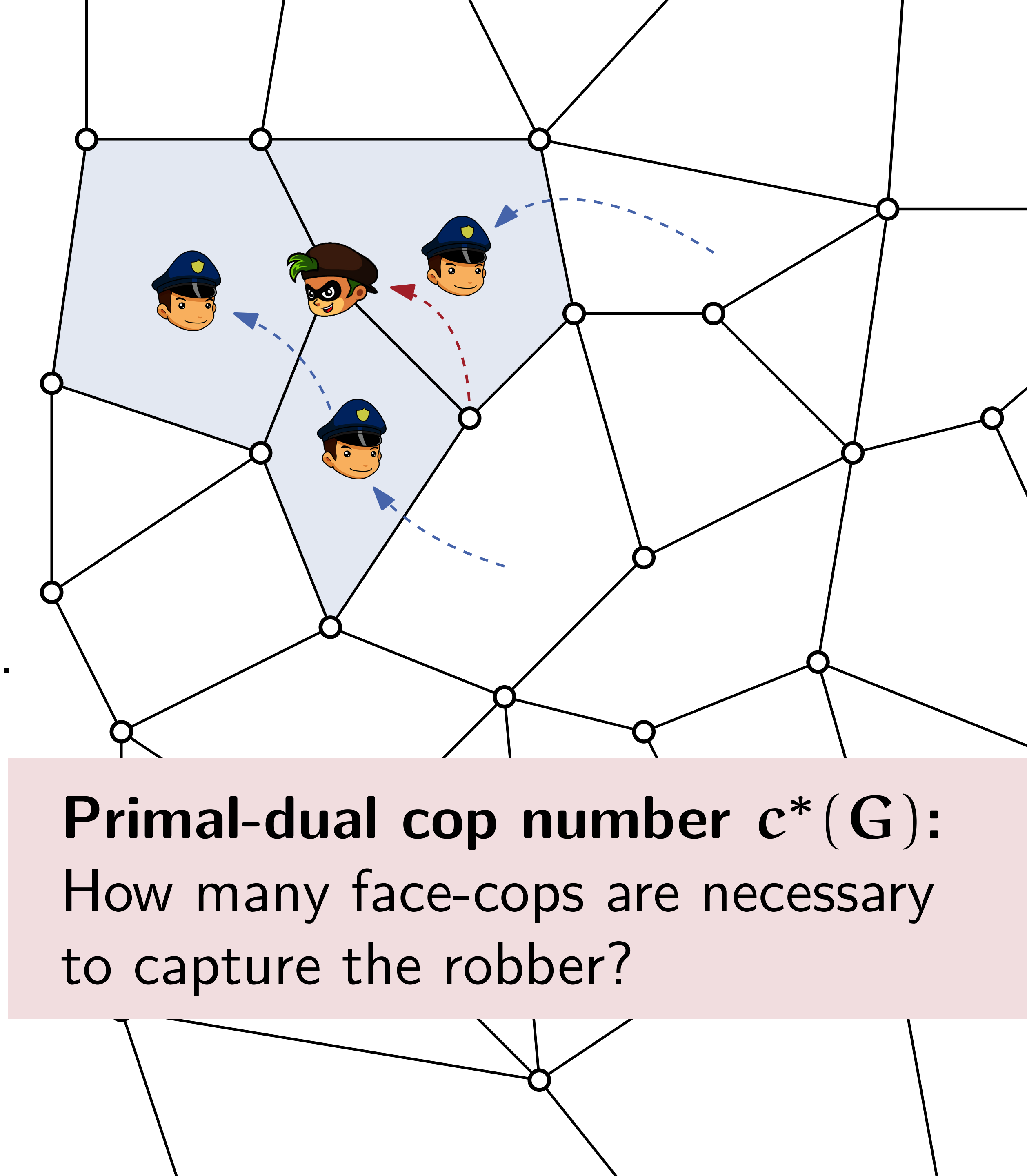


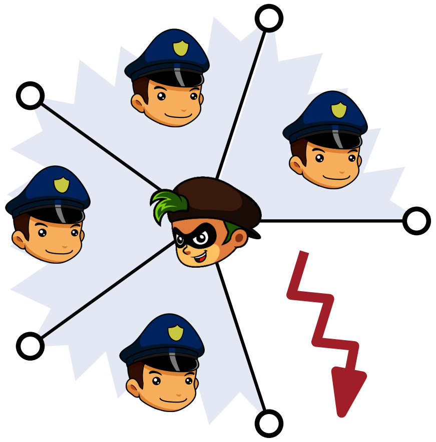
Primal-dual cop number $c^*(G)$:
How many face-cops are necessary to capture the robber?

Results

Trivial lower bound:

$c^*(G) \geqslant \Delta(G)$ (if G is 2-connected)

↳ max. degree



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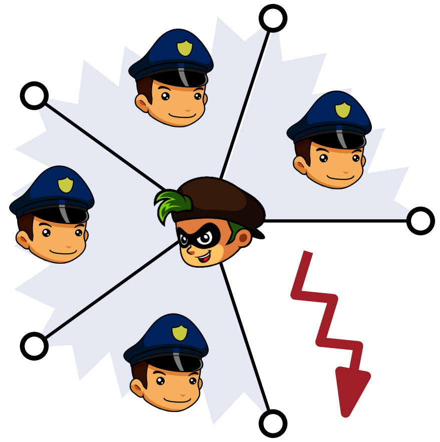

Question:

Is there an upper bound on $c^*(G)$ in terms of $\Delta(G)$?

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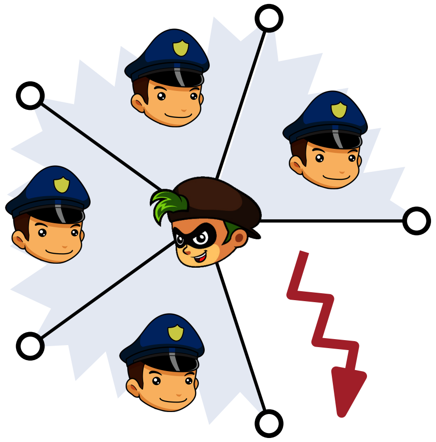


Question:

Is there an upper bound on $c^*(G)$ in terms of $\Delta(G)$?

Theorem:

For a plane graph G:

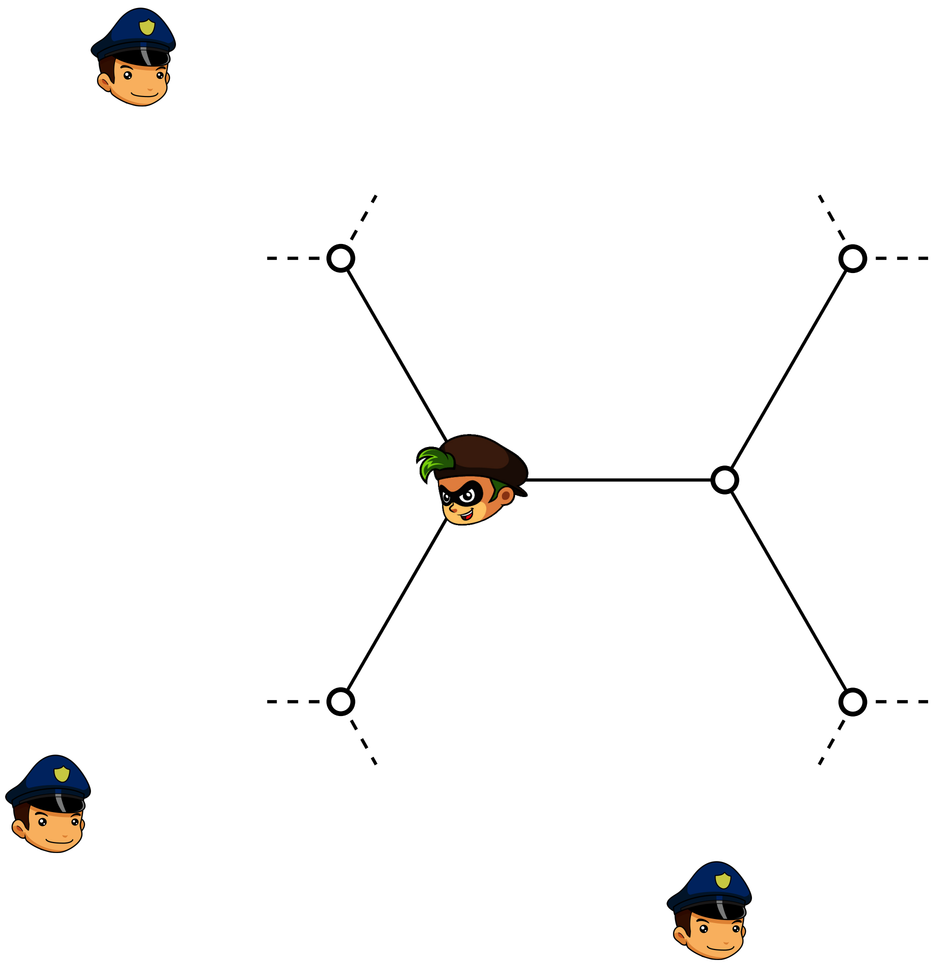
$c^*(G) \leqslant 3$ if $\Delta(G) \leqslant 3$

$c^*(G) \leqslant 12$ if $\Delta(G) \leqslant 4$

$c^*(G) \in \Omega\left(\sqrt{\log(n)}\right)$ if $\Delta(G) \geqslant 5$

$c^*(G) \leqslant 3$ for $\Delta(G) \leqslant 3$

(Simplification: 3-regular, 2-connected)



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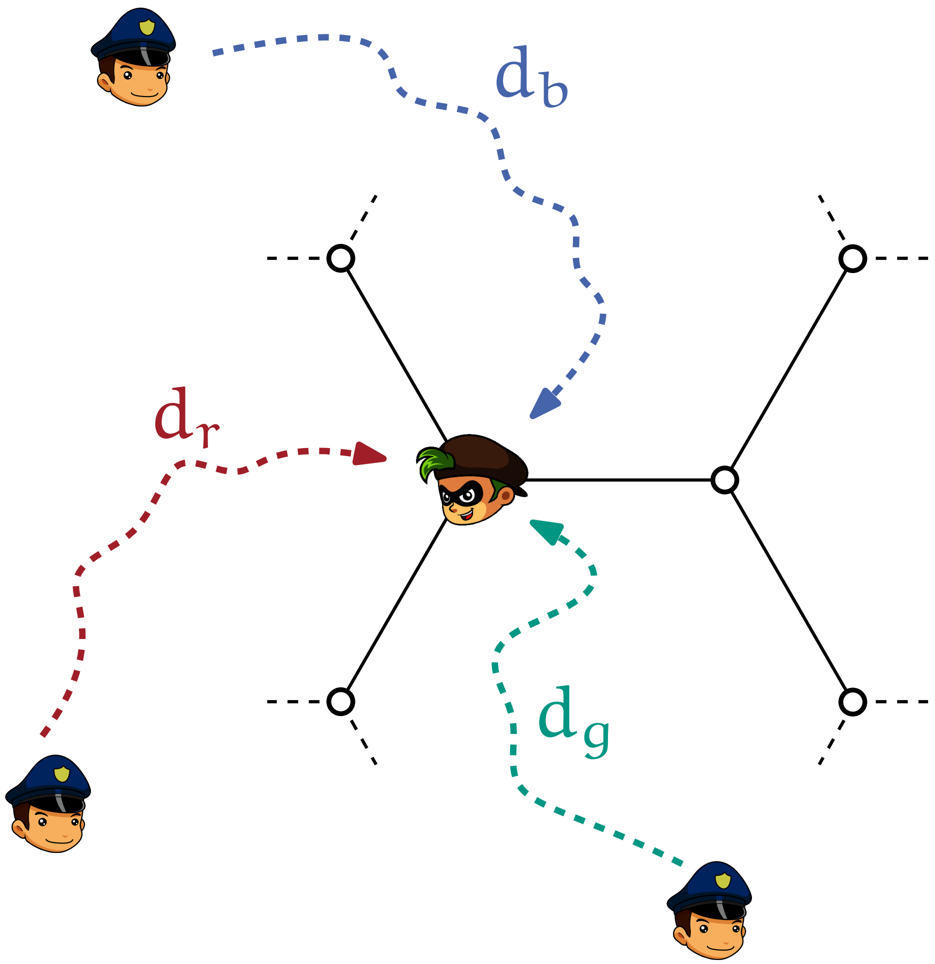

- Cops *choose* their target faces.

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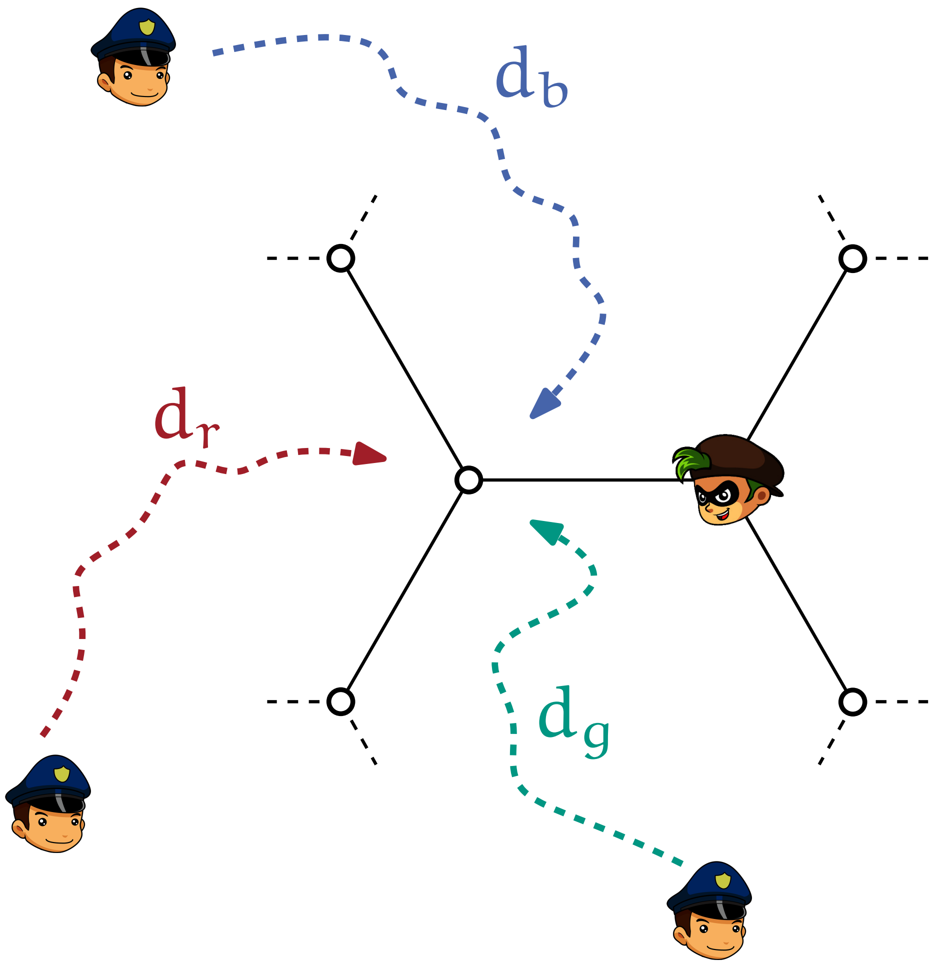

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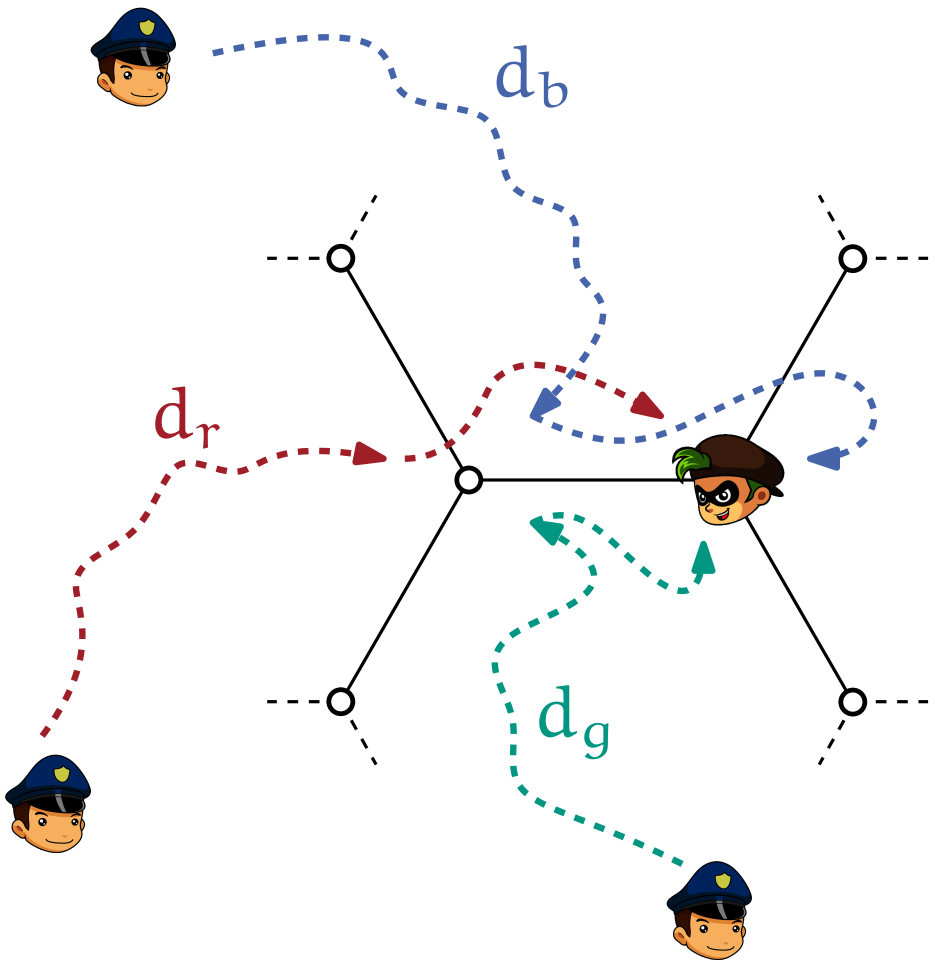

- Cops *choose* their target faces.
- Cops *update* their target faces.
  - d_b and d_r increased by 1
  - d_g did *not* change
  - each cop may take one step

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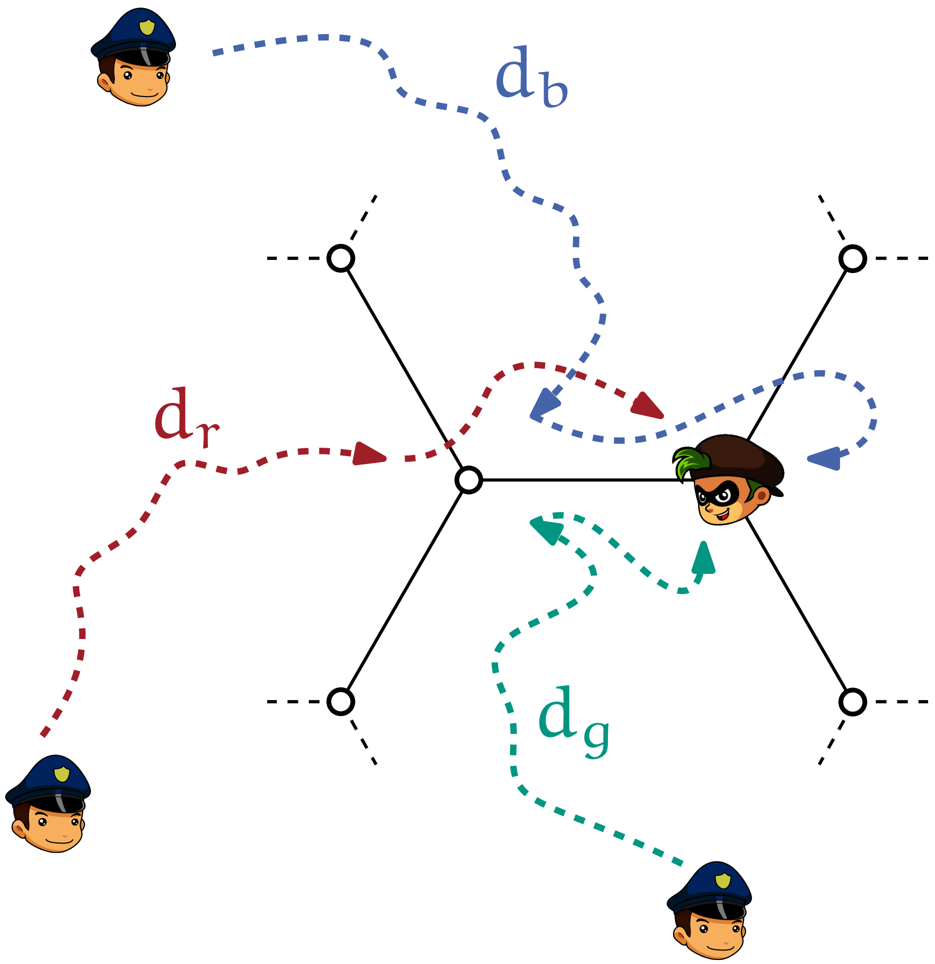

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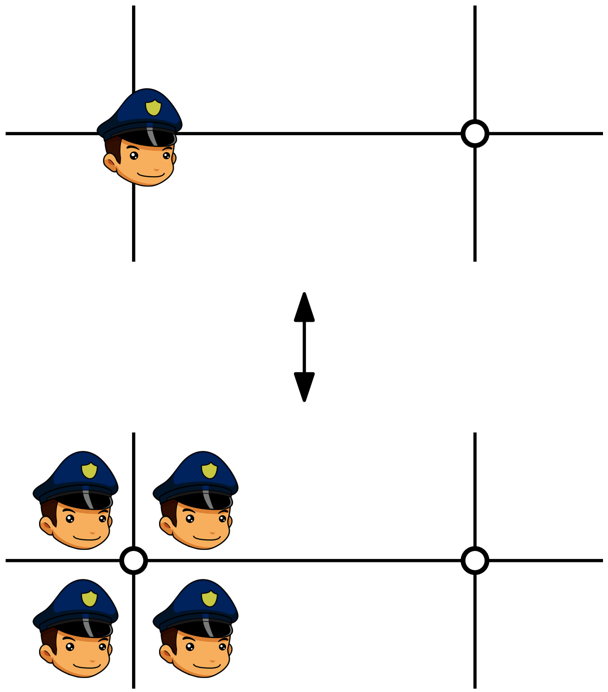
$\rightsquigarrow$ $d_b + d_r + d_g$ decreases during each cop-turn

(There is an edge case left in the endgame.)

$c^*(G) \leqslant 12$ for $\Delta(G) \leqslant 4$

(Simplification: 4-regular, 2-connected)

Idea: Four face-cops can simulate a vertex cop.



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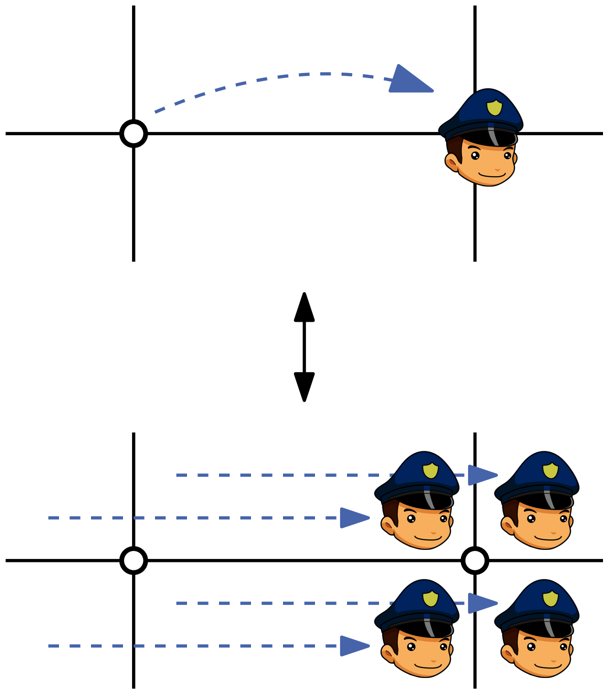

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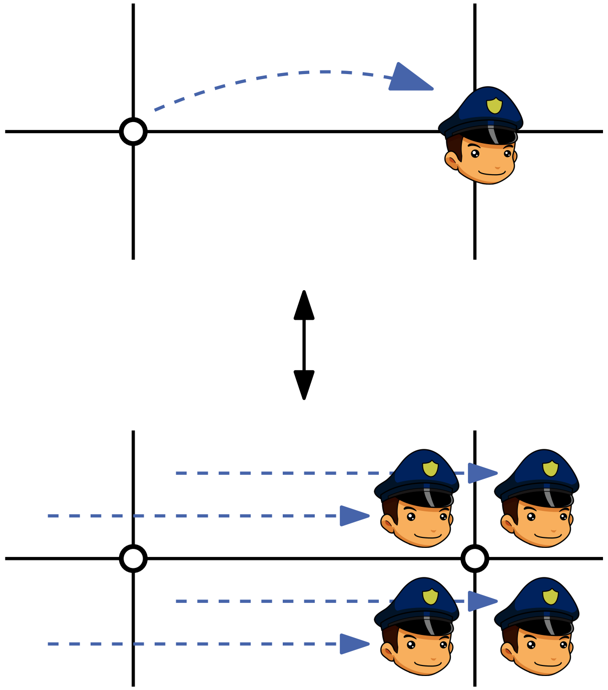


Theorem: (Aigner, Fromme 1984)
$c(G) \leqslant 3$ for all planar graphs G.

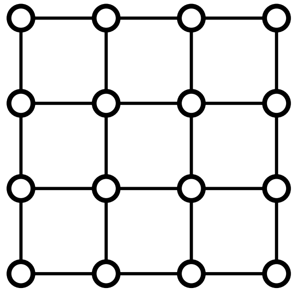
$\rightsquigarrow$

$4 \cdot 3 = 12$ face-cops always suffice

(in planar graphs with $\Delta(G) \leqslant 4$)

$c^*(G)$ is unbounded for $\Delta(G) \geqslant 5$

Theorem: (Nisse, Suchan 2008)
$c_{p,q}(G_{n,n}) \in \Omega\left(\sqrt{\log(n)}\right)$ for the $n \times n$-grid graph $G_{n,n}$.



variant of *classical* game

$p < q$: cop and robber speeds (edges per turn)

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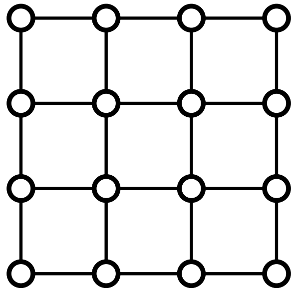

$G_{4,4}$

↑ variant of *classical* game

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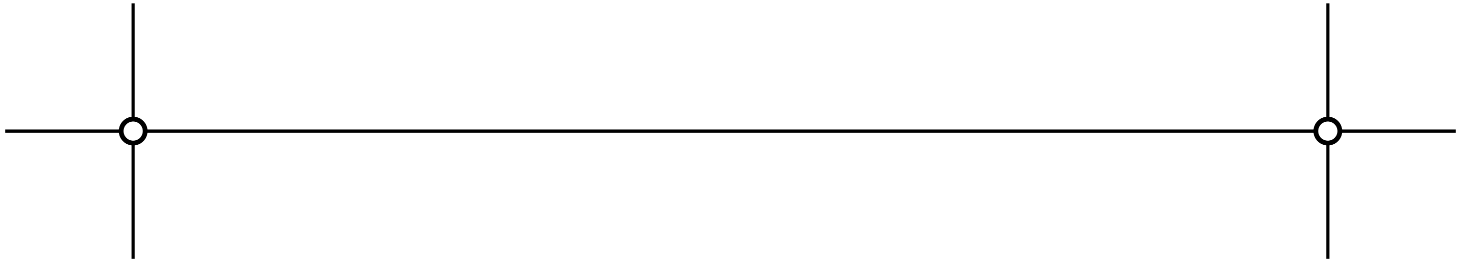
Idea: Simulate this in our primal-dual variant.

grid edge:



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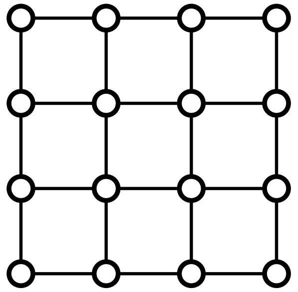

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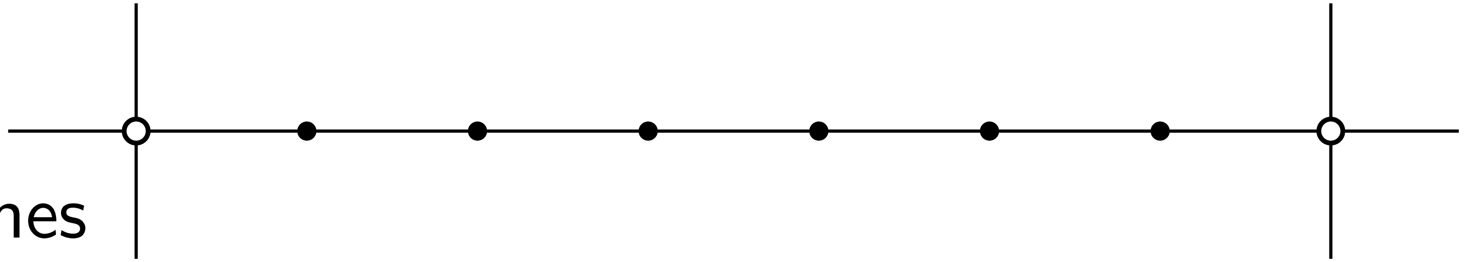

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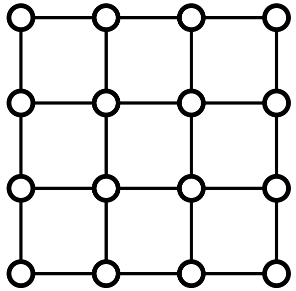

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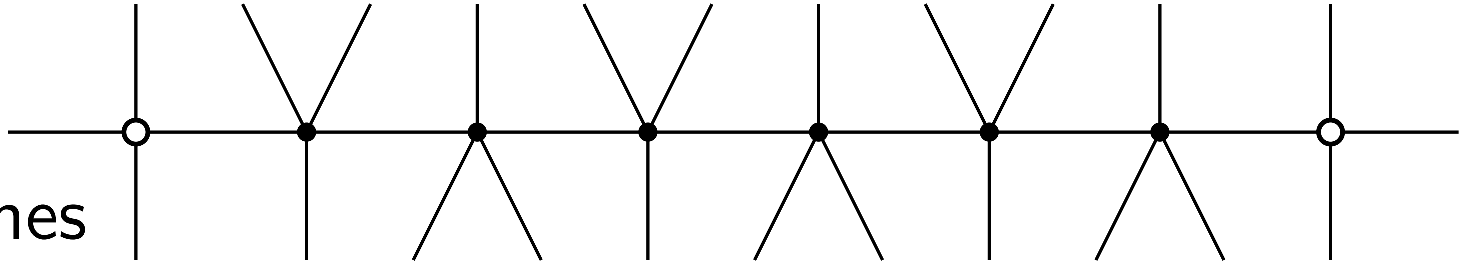

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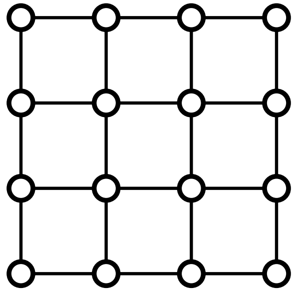
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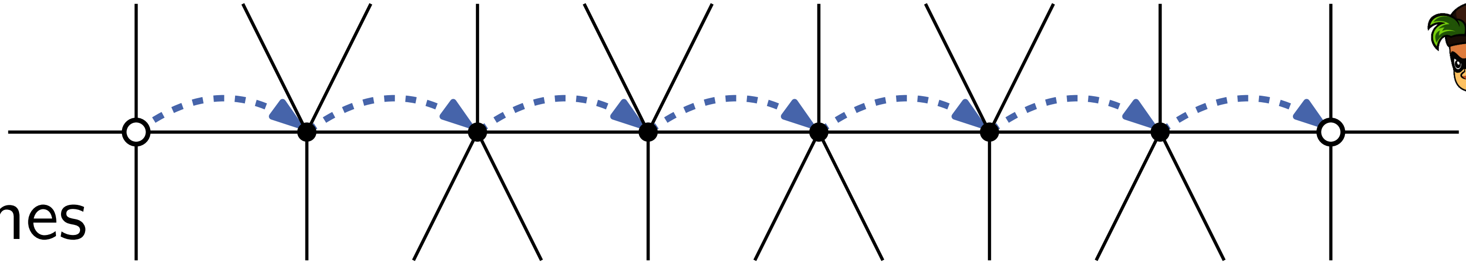


$h + 1$ steps

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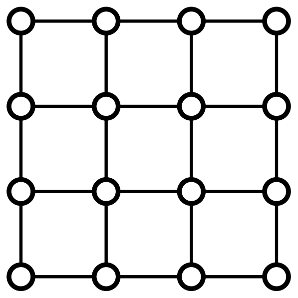

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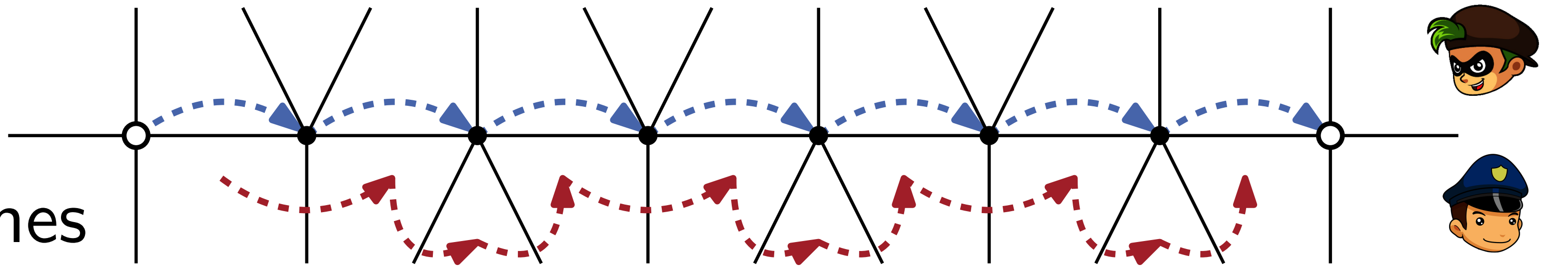

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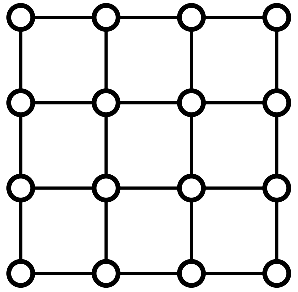
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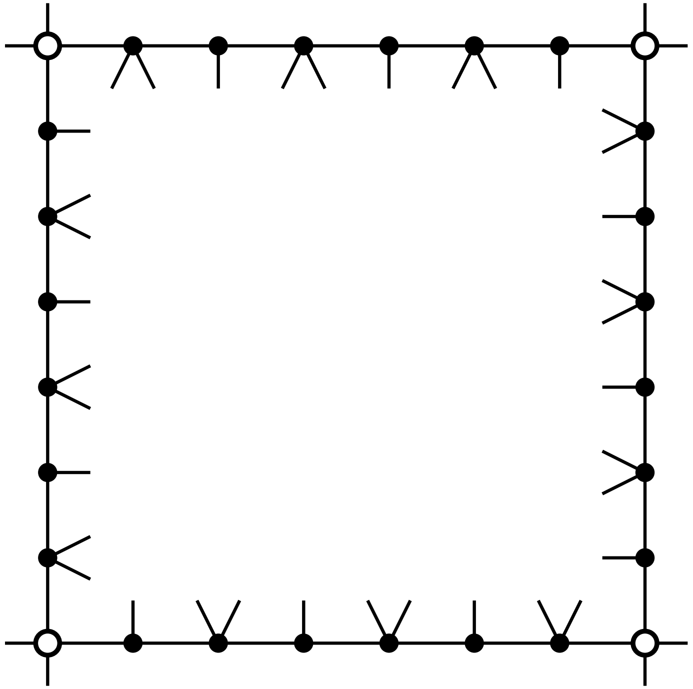

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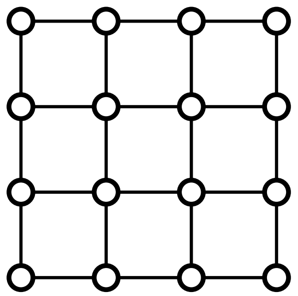

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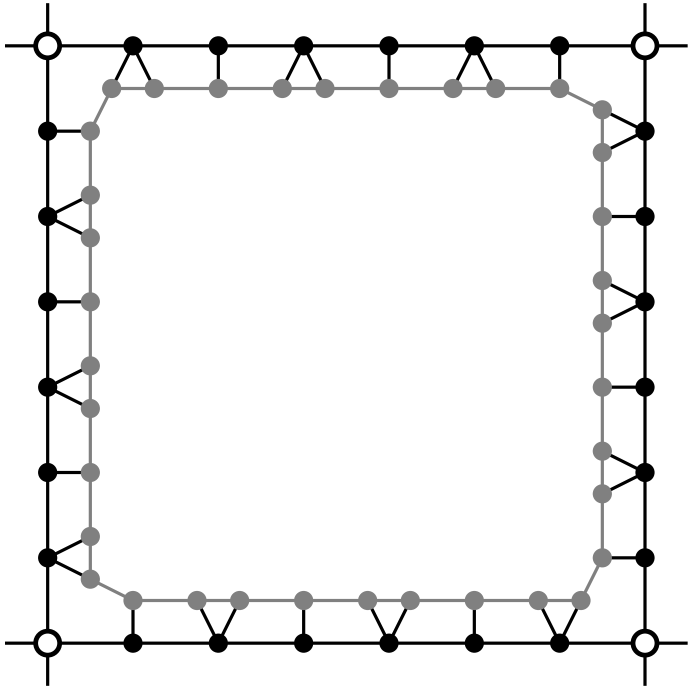

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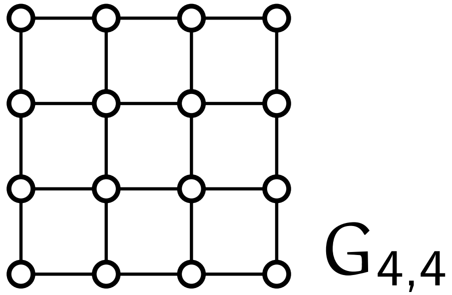

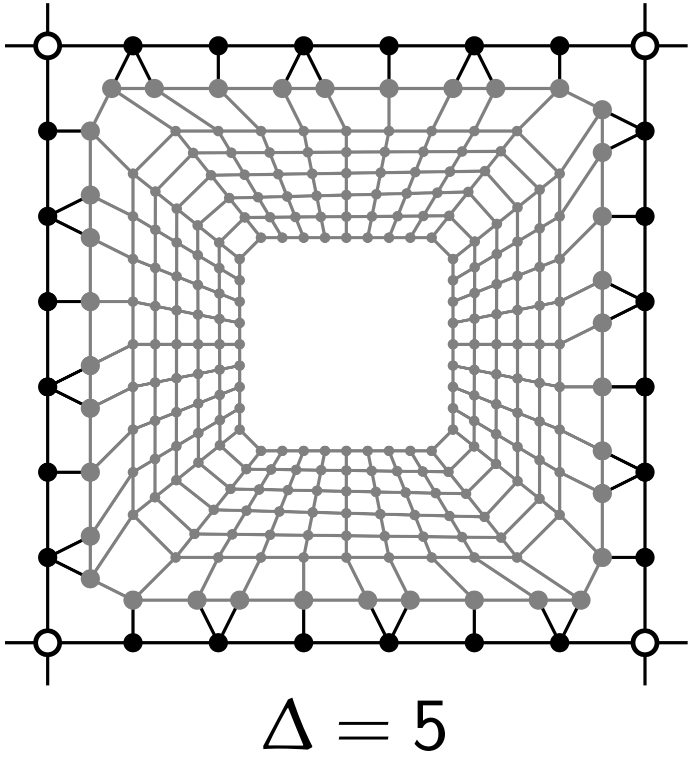

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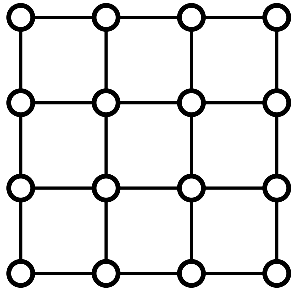

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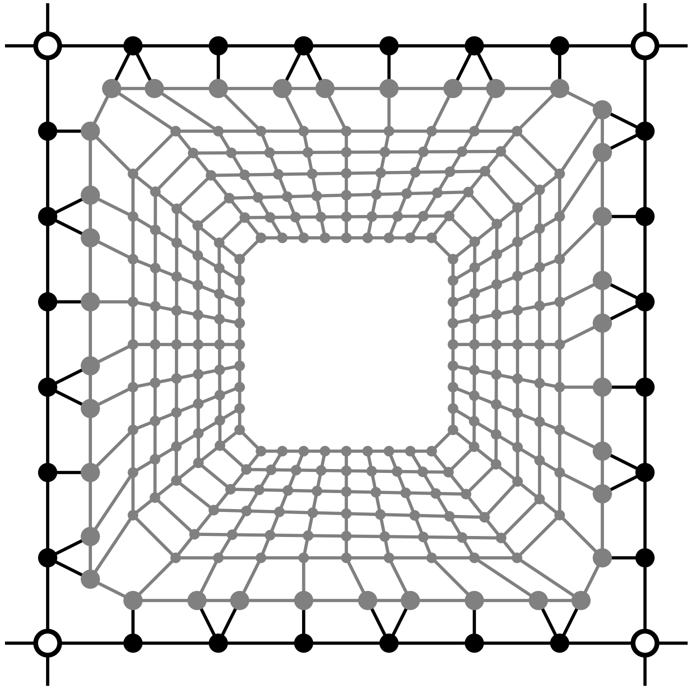

$\Delta = 5$

Robber strategy:

- copy strategy of Nisse and Suchan
  - robber moves between grid vertices "o"
  - "rounds" face-cop to nearest grid vertex
- inner rings $\rightsquigarrow$ no shortcuts for face-cops
- robber is faster $\rightsquigarrow$ $c^*(G) \in \Omega\left(\sqrt{\log(n)}\right)$

Open Problems

Problem 1

Find exact bounds:

✓ $\Delta(G) \leqslant 3$: $c^*(G) = 3$

? $\Delta(G) \leqslant 4$: $4 \leqslant c^*(G) \leqslant 12$

? $\Delta(G) \leqslant 5$: $c^*(G) \in \Omega\left(\sqrt{\log(n)}\right)$

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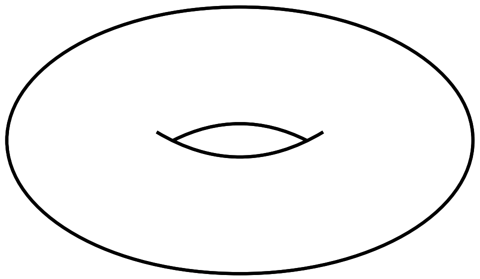
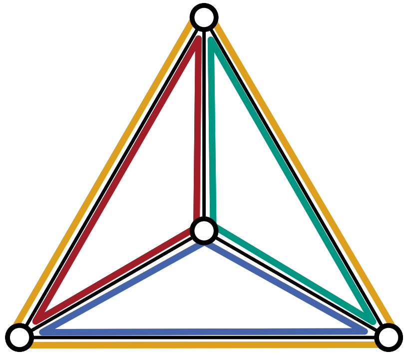
Problem 2

Generalize the game:





- consider graphs with crossing-free embeddings on other surfaces
- use cycle double cover instead of faces