Recognizing Unit Disk Graphs in Hyperbolic Geometry is $\exists_R$-Complete

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Definition:
$G$ is a (Euclidean) unit disk graph if it is the intersection graph of unit disks in $\mathbb{R}^2$.

UDG: Class of unit disk graphs.
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**Why unit disk?**

- all disks must have the same radius $r$
  $\Rightarrow$ *equally sized* disk graphs
- scaling the plane $\mathbb{R}^2$ allows to assume $r = 1$
Euclidean vs. Hyperbolic Geometry

Formalized by axiomatic systems:
(Euclid, Hilbert, …)
Euclidean vs. Hyperbolic Geometry

Formalized by axiomatic systems:
(Euclid, Hilbert, ...)

- Euclidean plane $\mathbb{R}^2$
- Hyperbolic plane $\mathbb{H}^2$

- Incidence
- Order
- Congruence
- Continuity
- Parallels
- $\neg$ Parallels

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Euclidean plane $\mathbb{R}^2$

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absolute geometry
Euclidean vs. Hyperbolic Geometry

Formalized by axiomatic systems: (Euclid, Hilbert, ...)

Models for $\mathbb{H}^2$:
- embedd $\mathbb{H}^2$ into $\mathbb{R}^d$
  $\sim$ allows to use human intuition for $\mathbb{R}^2$ in $\mathbb{H}^2$
- many different options:
  - Beltrami-Klein model
  - Poincaré model
  - Hyperboloid model
Hyperbolic Unit Disk Graphs

**Definition:**
A graph is a *hyperbolic unit disk graph* if it is the intersection graph of *equally sized* disks in $\mathbb{H}^2$.

**HUDG:** Class of hyperbolic unit disk graphs.
Hyperbolic Unit Disk Graphs

Definition:
A graph is a hyperbolic unit disk graph if it is the intersection graph of equally sized disks in $\mathbb{H}^2$.

HUDG: Class of hyperbolic unit disk graphs.

G: star with seven leaves
G $\notin$ UDG
G $\in$ HUDG

Poincaré disk model:
- $\mathbb{H}^2 \simeq$ interior of a disk
- circles $\sim$ circles
- closer to the boundary: more distorted/compressed
- $\sim$ all circles have equal area
Our Results

**Theorem:**
Recognizing hyperbolic unit disk graphs is $\exists \mathbb{R}$-complete.
Our Results

**Theorem:**
Recognizing hyperbolic unit disk graphs is $\exists R$-complete.

**Complexity class $\exists R$:**
All problems reducible to the existential theory of the reals (ETR).

Decide truth of formulas like:

$$\exists x_1, \ldots, x_n \in \mathbb{R}^n : x_1x_2 + 3x_3 = 10 \land x_2x_4 \leq 1$$

(polynomial systems of equations and inequalities)
\( \exists R \)-Hardness

Simple Stretchability

- every two lines intersect
- no more than two lines intersect in any point

pseudolines

lines in \( \mathbb{R}^2 \)
\( \exists R \)-Hardness

Simple Stretchability

- every two lines intersect
- no more than two lines intersect in any point

\textbf{Theorem:} (Mnëv 1988)
Simple Stretchability is \( \exists R \)-complete.
**\( \exists \mathbb{R} \)-Hardness**

**Simple Stretchability**
- Every two lines intersect
- No more than two lines intersect in any point

![Diagram of pseudolines and lines in \( \mathbb{R}^2 \)]

**Theorem:** (Mnëv 1988)
Simple Stretchability is \( \exists \mathbb{R} \)-complete.

**Theorem:** (McDiarmid, Müller 2010)
Simple Stretchability can be reduced to recognizing UDGs.
∃R-Hardness

D - Instance of Simple Stretchability
G_D - Graph constructed from D following McDiarmid and Müller

\[ \text{D stretchable in } \mathbb{R}^2 \iff G_D \in \text{UDG} \]

[McDiarmid, Müller 2010]
∃R-Hardness

\( D \) - Instance of Simple Stretchability

\( G_D \) - Graph constructed from \( D \) following McDiarmid and Müller

\[ \text{D stretchable in } \mathbb{R}^2 \iff G_D \in \text{UDG} \]

\[ \implies G_D \in \text{HUDG} \]

[McDiarmid, Müller 2010]

Theorem: (Bläsius, Friedrich, Katzmann, Stephan 2023)
It holds that \( \text{UDG} \subseteq \text{HUDG} \).
∃R-Hardness

\( D \) - Instance of Simple Stretchability

\( G_D \) - Graph constructed from \( D \) following McDiarmid and Müller

\[
D \text{ stretchable in } \mathbb{R}^2 \iff G_D \in UDG
\]

\[
D \text{ stretchable in } \mathbb{H}^2 \iff G_D \in HUDG
\]

Theorem: (Bläsius, Friedrich, Katzmann, Stephan 2023)
It holds that \( UDG \subseteq HUDG \).

same argument:
uses only absolute geometry
\( \exists R \)-Hardness

\( D \) - Instance of Simple Stretchability

\( G_D \) - Graph constructed from \( D \) following McDiarmid and Müller

\( D \) stretchable in \( \mathbb{R}^2 \)

\( G_D \in \text{UDG} \)

\( H^2 \)

\( G_D \in \text{HUDG} \)

\[ \text{Theorem: (Bläsius, Friedrich, Katzmann, Stephan 2023)} \]

\[ \text{It holds that UDG} \subseteq \text{HUDG}. \]
Simple Stretchability

**Beltrami-Klein model:**

- $\mathbb{H}^2 \cong$ interior of disk $D$
- hyperbolic lines $\cong$ chords of $D$
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**Theorem:**
$D$ stretchable in $\mathbb{R}^2 \iff D$ stretchable in $\mathbb{H}^2$
\( \exists R \)-Membership

**Idea:** Given coordinates, verify that all neighbors are closer to each other than all non-neighbors. (in polynomial time on a real RAM machine)

**Problem:** Involves computing distances in \( \mathbb{H}^2 \): requires hyperbolic functions \( \sim \) not computable on a real RAM
∃R-Membership

Idea: Given coordinates, verify that all neighbors are closer to each other than all non-neighbors.

Problem: Involves computing distances in \( \mathbb{H}^2 \):
requires hyperbolic functions \( \leadsto \) not computable on a real RAM

Hyperboloid model:
- \( \mathbb{H}^2 \cong \) points in \( \mathbb{R}^3 \) with \( z^2 - x^2 - y^2 = 1 \) and \( z > 0 \)
- \( d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \text{arcosh}(z_1z_2 - x_1x_2 - y_1y_2) \)
∃R-Membership

**Idea:** Given coordinates, verify that all neighbors are closer to each other than all non-neighbors.

**Problem:** Involves computing distances in $\mathbb{H}^2$:
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**Hyperboloid model:**
- $\mathbb{H}^2 \cong$ points in $\mathbb{R}^3$ with $z^2 - x^2 - y^2 = 1$ and $z > 0$
- $d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \text{arcosh}(z_1z_2 - x_1x_2 - y_1y_2)$
  - monotone function
  - polynomial
  - requires hyperbolic functions; not computable on a real RAM
Open Problems

Problem 1:
Generalize to higher dimensions:

Simple Stretchability of hyperplanes is $\exists R$-complete in $\mathbb{R}^d$.
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Generalize to higher dimensions:

Simple Stretchability of hyperplanes is $\exists \mathbb{R}$-complete in $\mathbb{R}^d$.

Problem 2:
Use reduction as a framework for more problems:

(Unit) Segment Graphs
Linkage Realization
RAC-Drawings