The Complexity of the Hausdorff Distance

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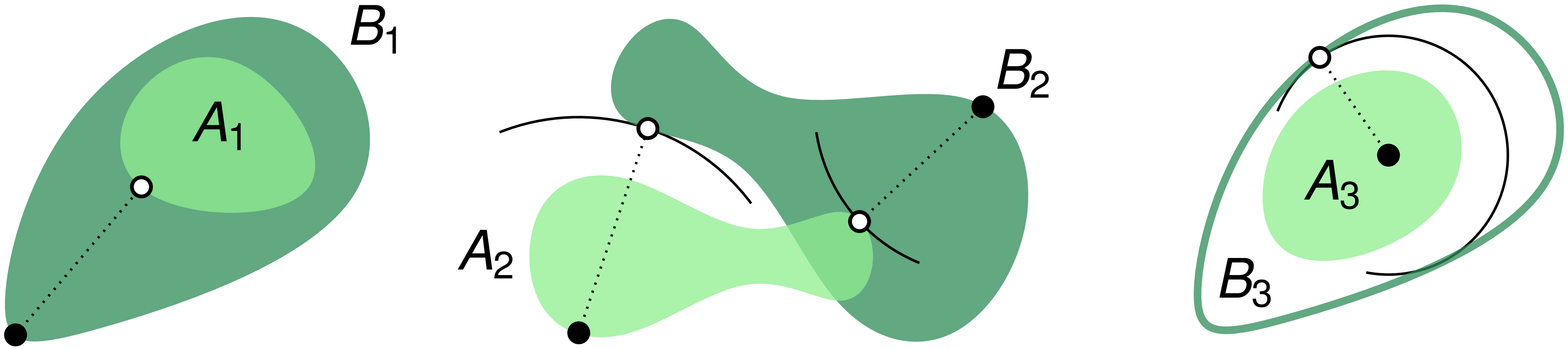
Paul Jungeblut, Linda Kleist, Till Miltzow



Hausdorff Distance: How similar are two sets?

HAUSDORFF

Given sets $A, B \subseteq \mathbb{R}^n$ and a rational number $t \in \mathbb{Q}$.

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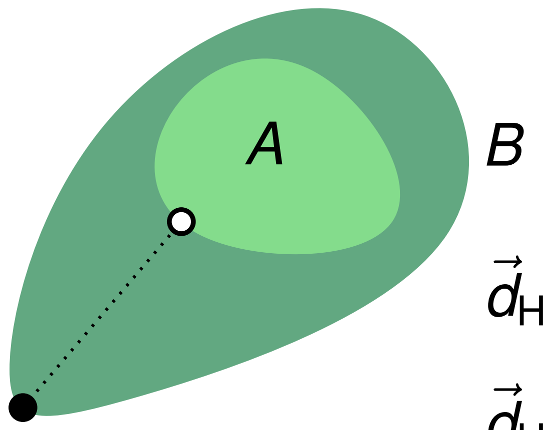

$\vec{d}_\mathrm{H}(A, B) = 0$

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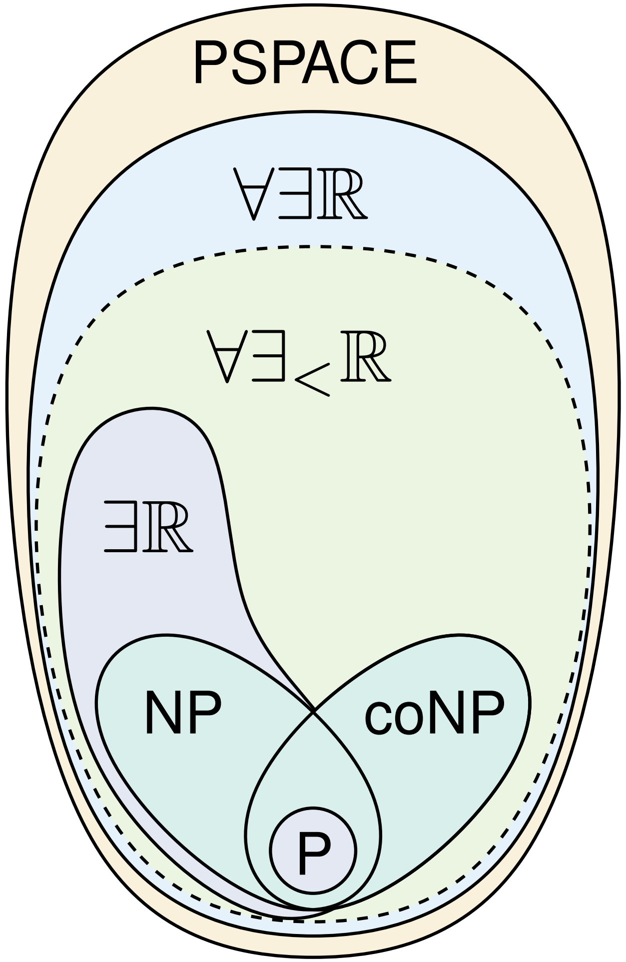
Main Result

Theorem:
Deciding whether $d_{\mathrm{H}}(A, B) \leq t$ is $\forall\exists_{<}\mathbb{R}$-complete.



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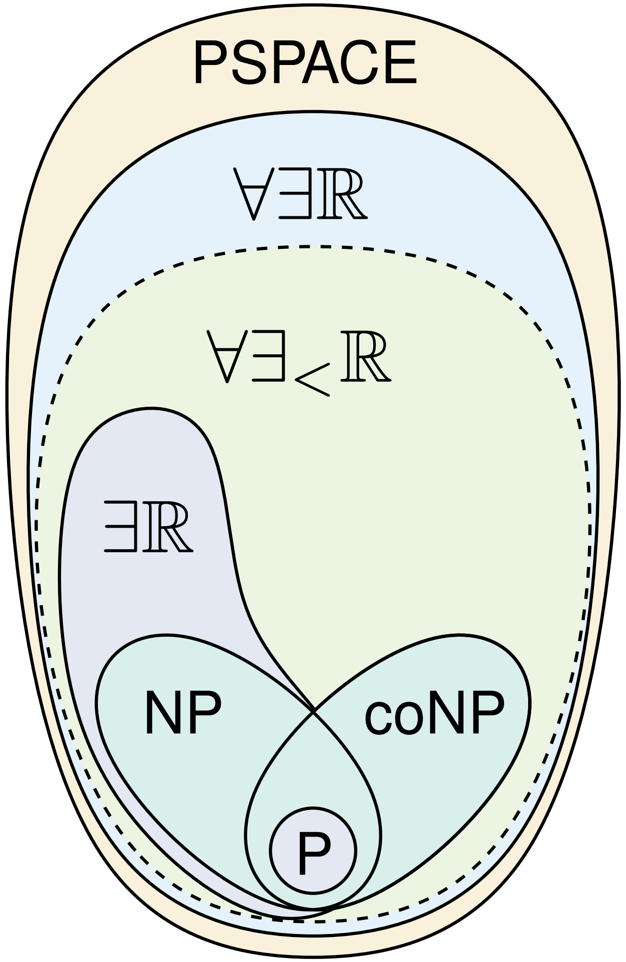
$A, B \subseteq \mathbb{R}^n$ are described by formulas with polynomial equations and inequalities:



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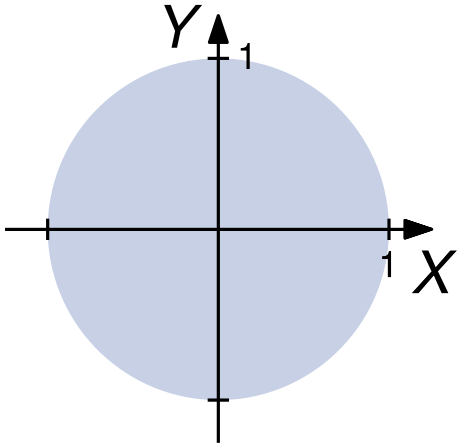
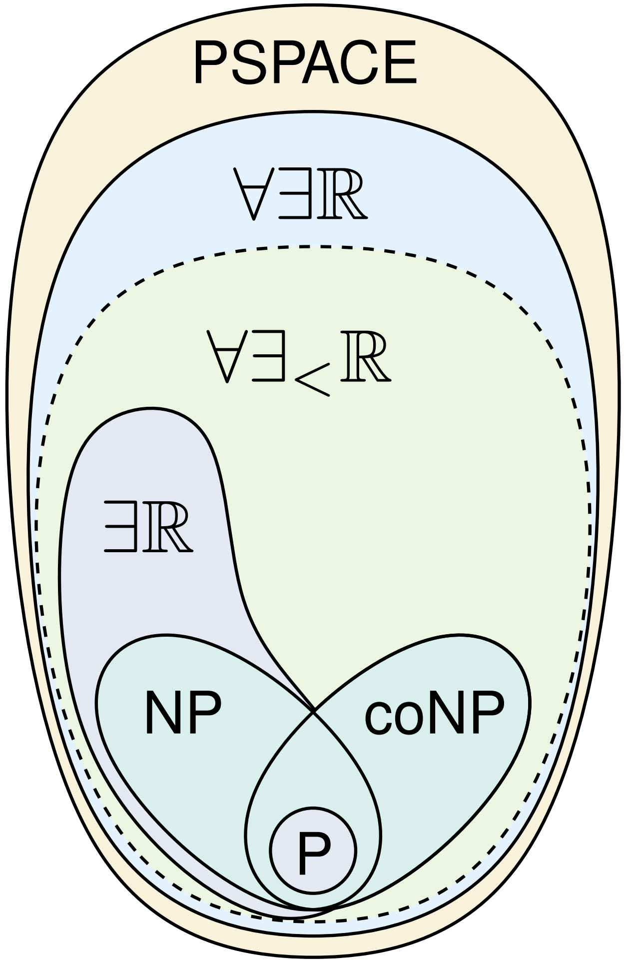
$\Phi(X, Y) :\equiv \quad X^2 + Y^2 \leq 1$





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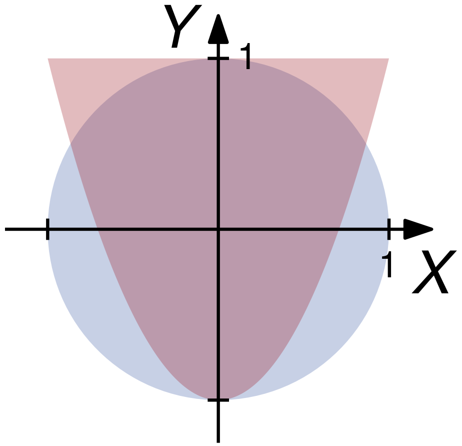

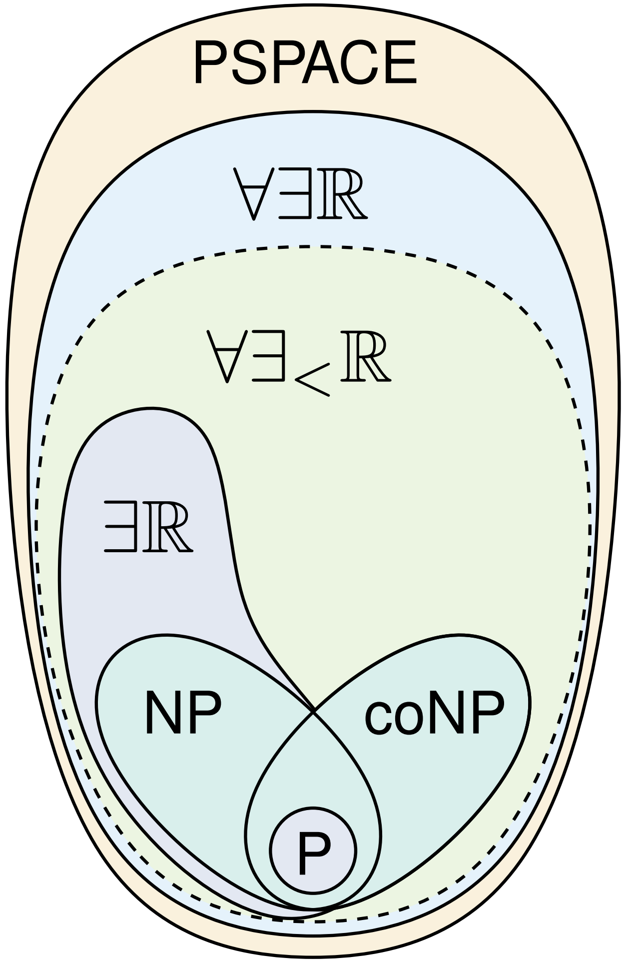

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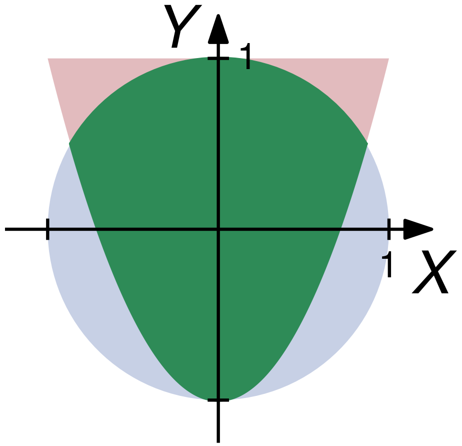

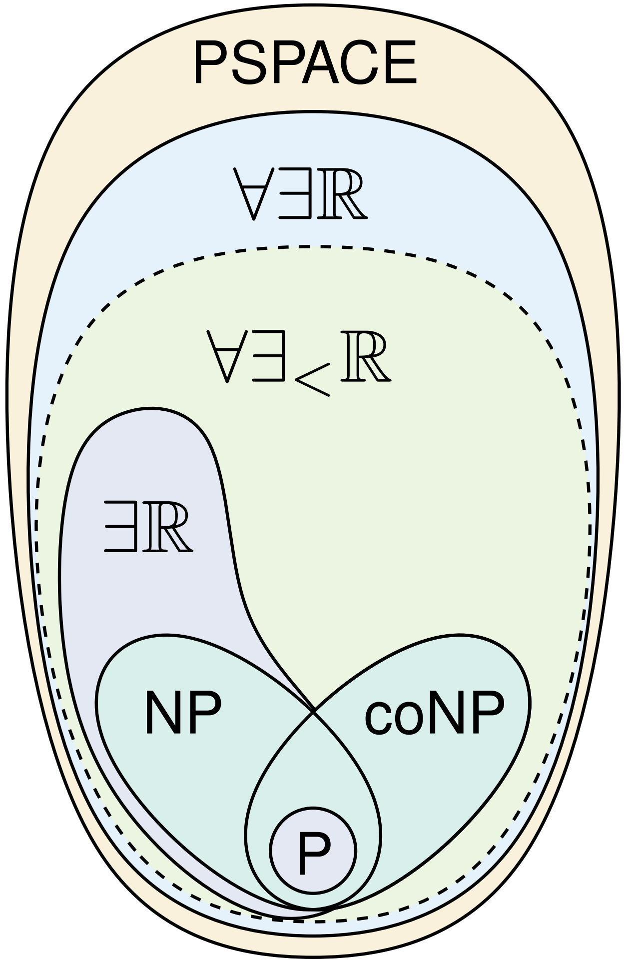

Universal Existential Theory of the Reals (UETR)

Definition:
The *universal existential theory of the reals* (UETR) consists of all true sentences of the form

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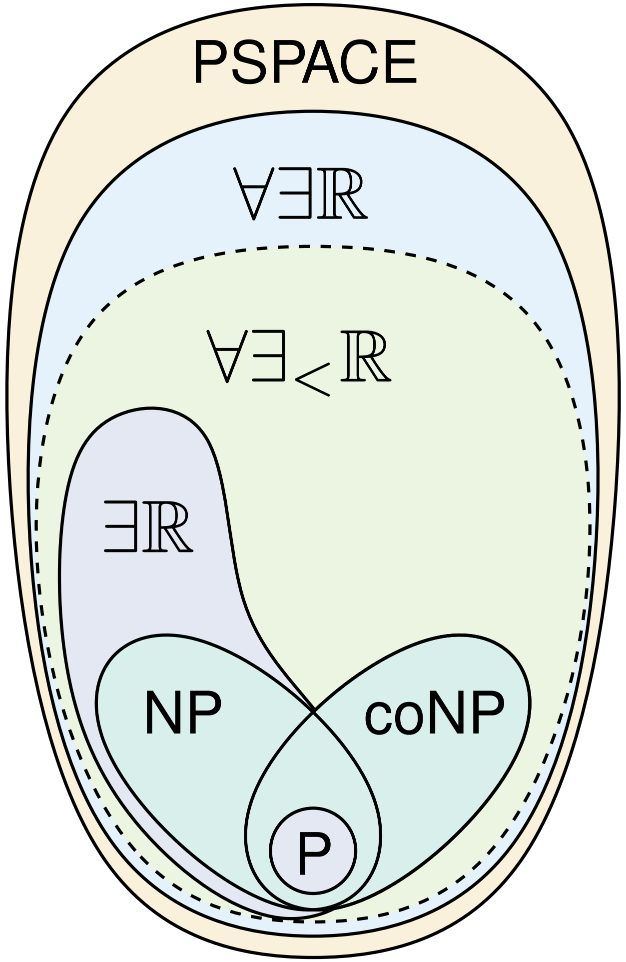
Examples:

- $\forall X \in \mathbb{R} . \exists Y \in \mathbb{R} : XY = 1$ (false: for $X = 0$ there is no Y)
- $\forall X \in \mathbb{R} . \exists Y \in \mathbb{R} : XY = 1 \vee X = 0$ (true)

Complexity Classes $\forall\exists\mathbb{R}$ and $\forall\exists_{<}\mathbb{R}$



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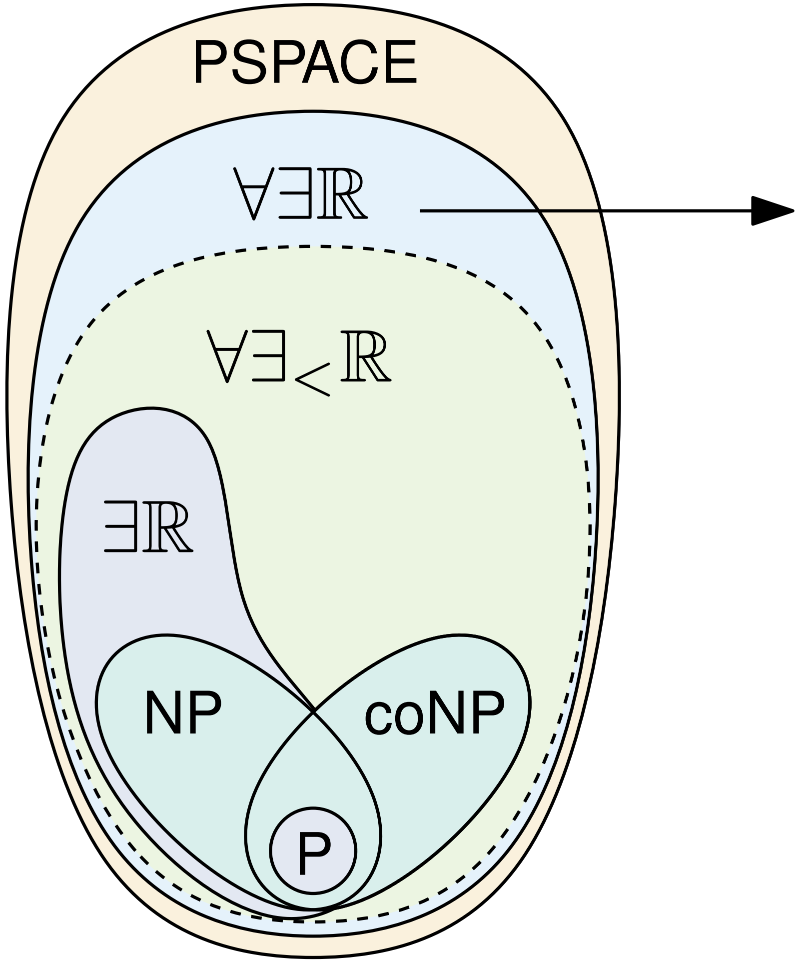

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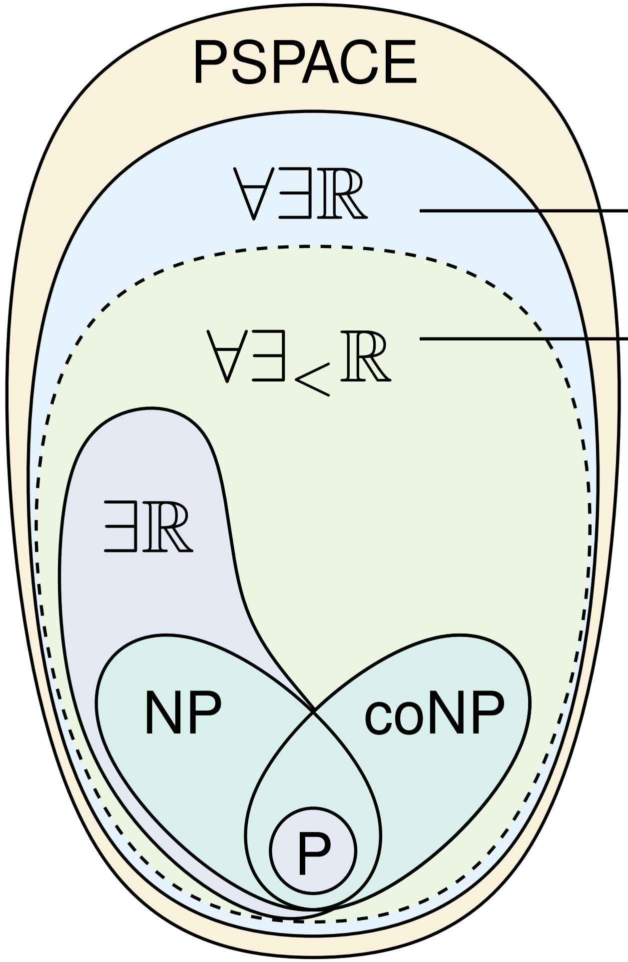

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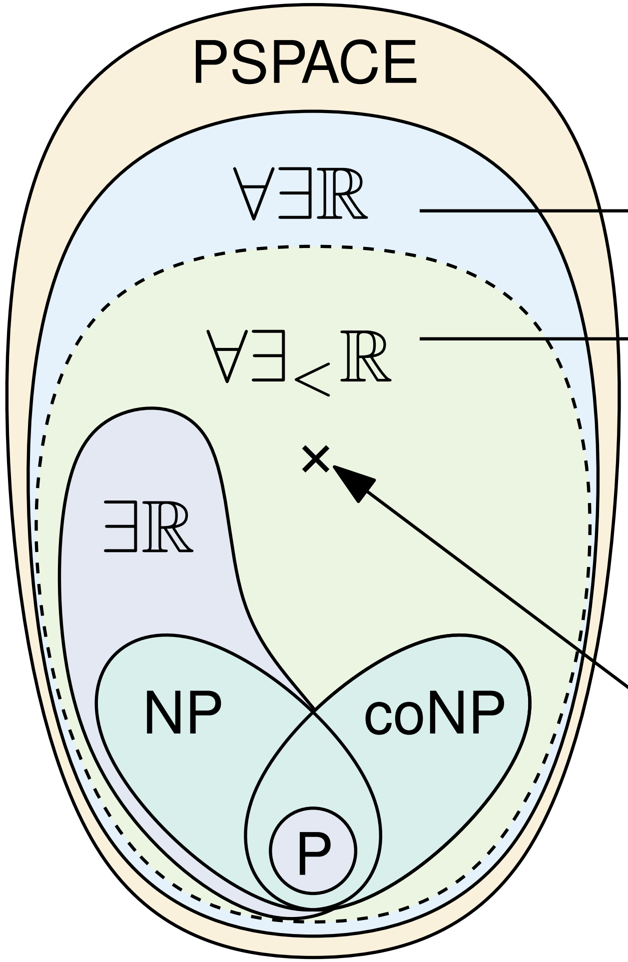

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Reduction: $A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x, Y)\}$

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Idea: Φ true $\rightsquigarrow$ $A = \mathbb{R}^n$ $\rightsquigarrow$ $d_H(A, B) = 0$

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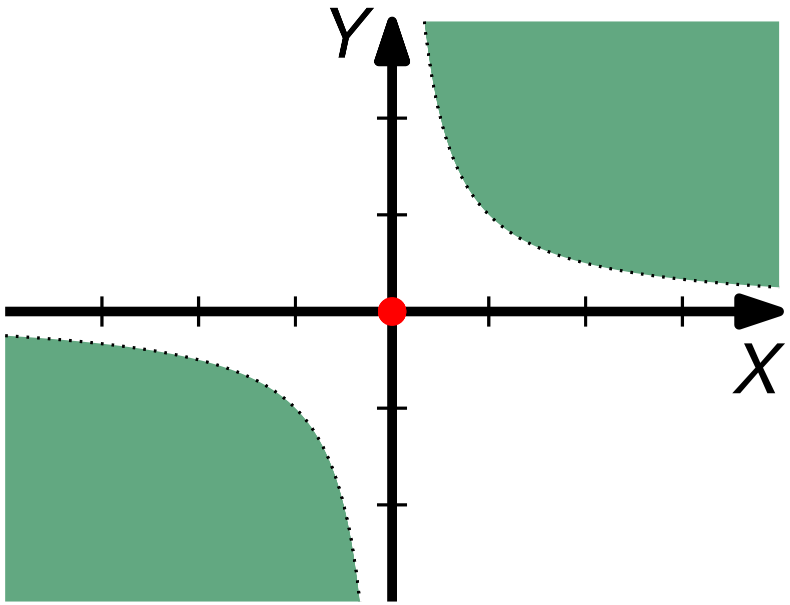
Problem 2:
Definition of A contains an $\exists$.

Problem 1: Blowing Up the Counterexamples

$$\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$$



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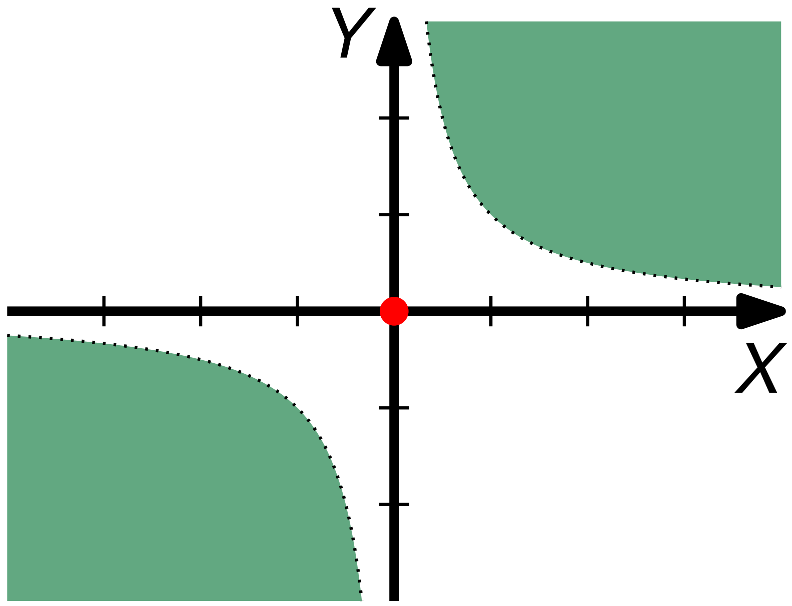

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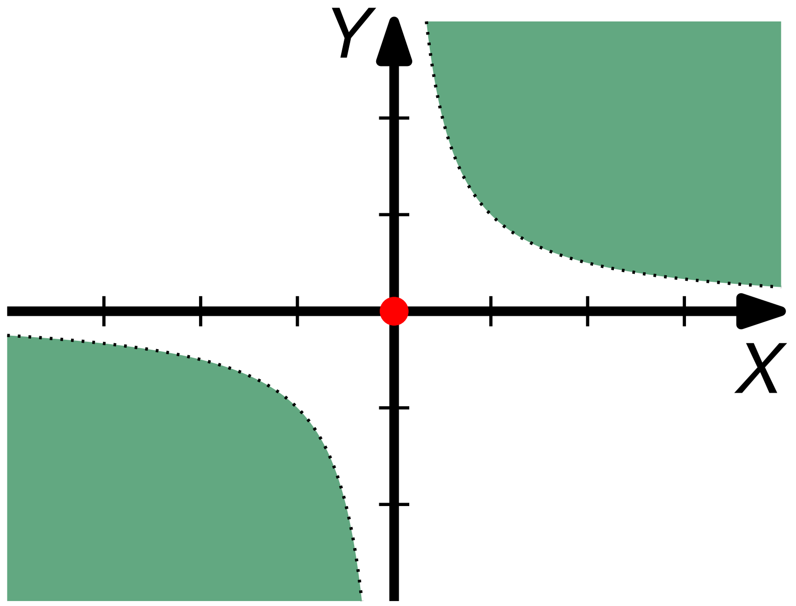

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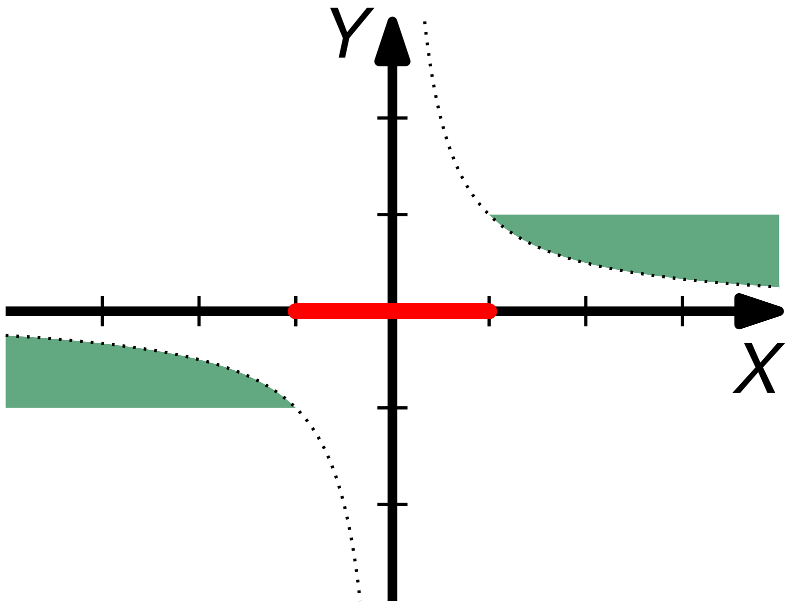

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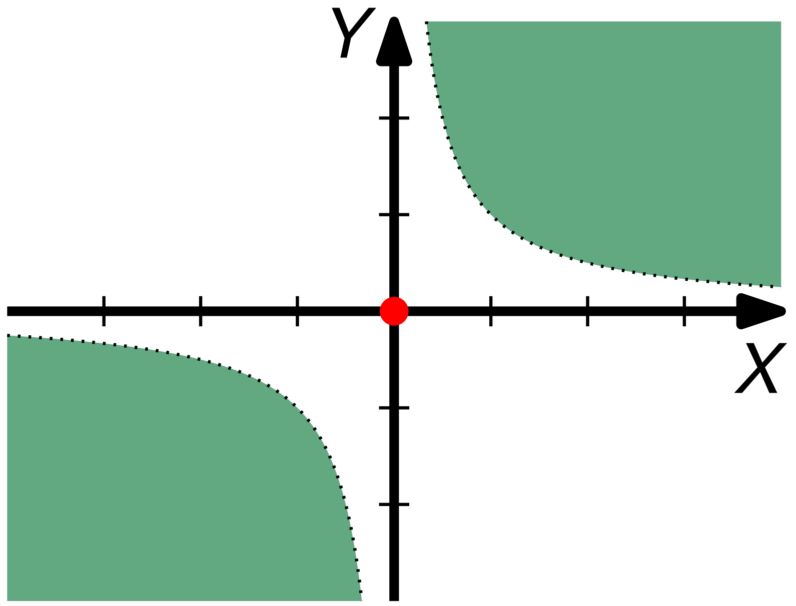

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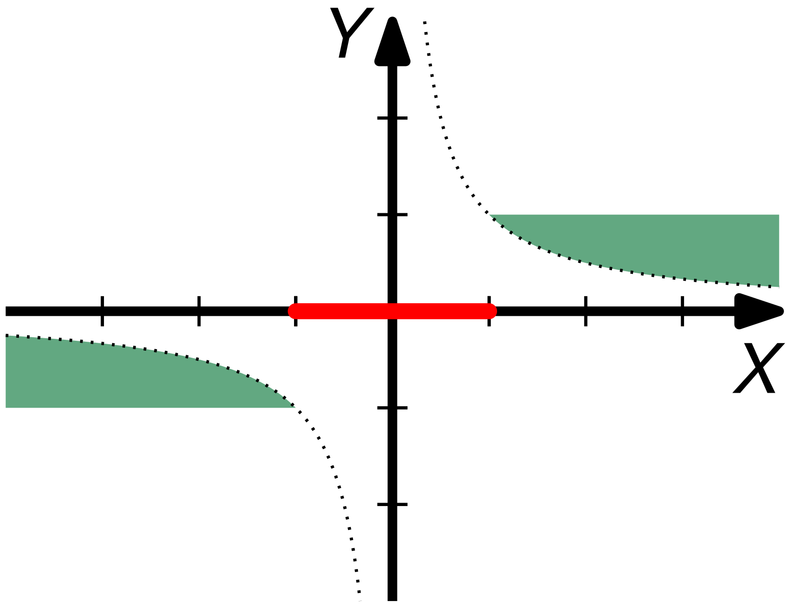

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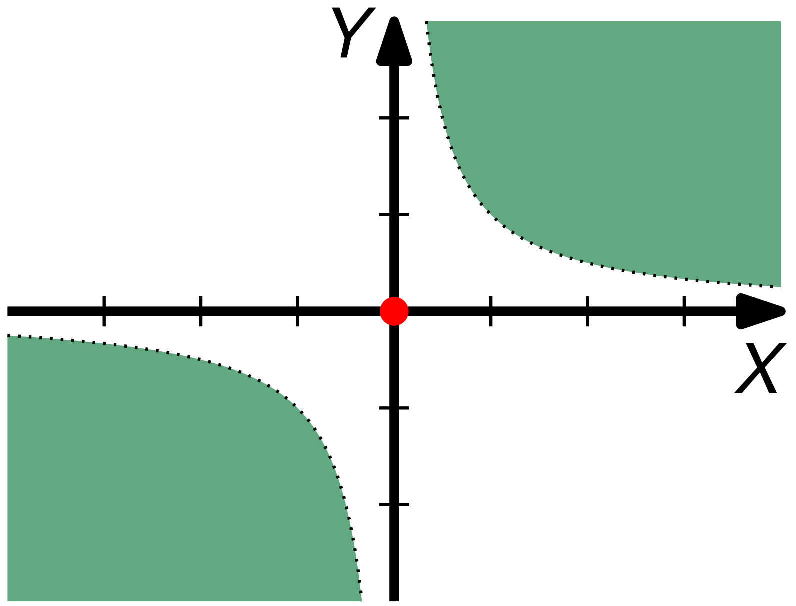

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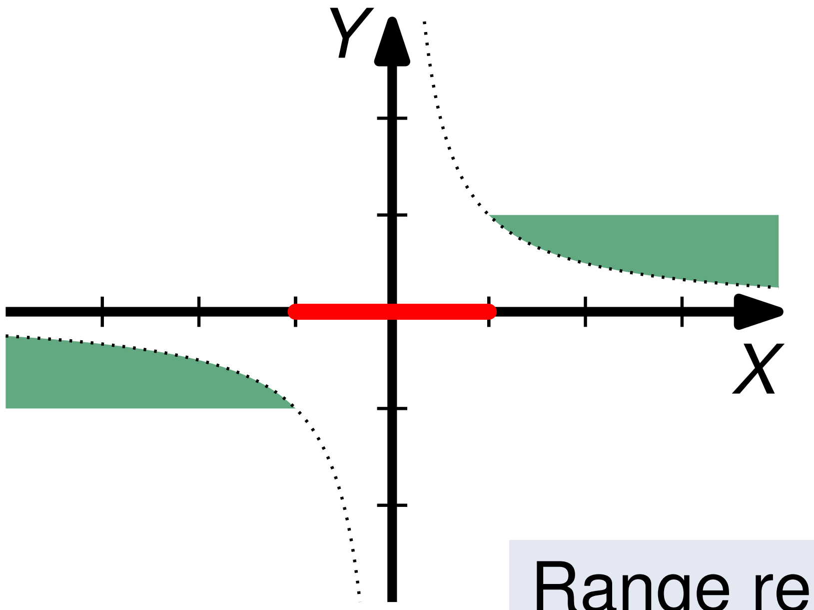

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Range restrictions only work with strict inequalities.
$\rightsquigarrow \forall\exists_{<}\mathbb{R}$-hardness

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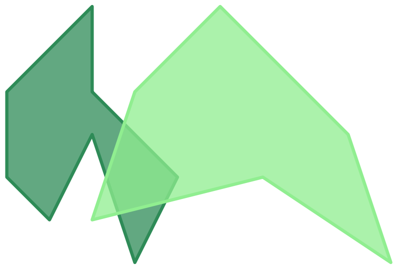
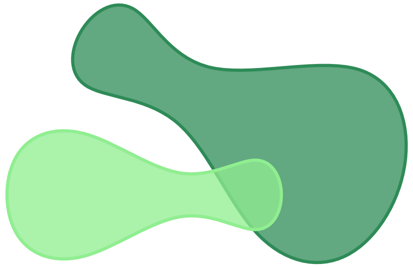


polygons $\rightsquigarrow$ P
[Alt et al. 1995]



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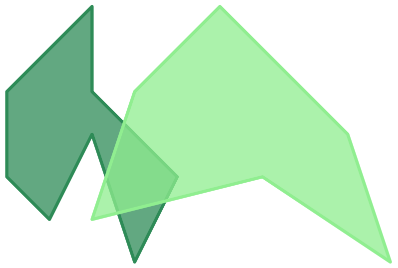

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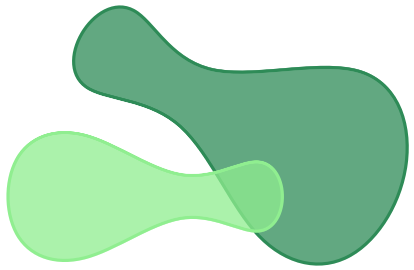

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