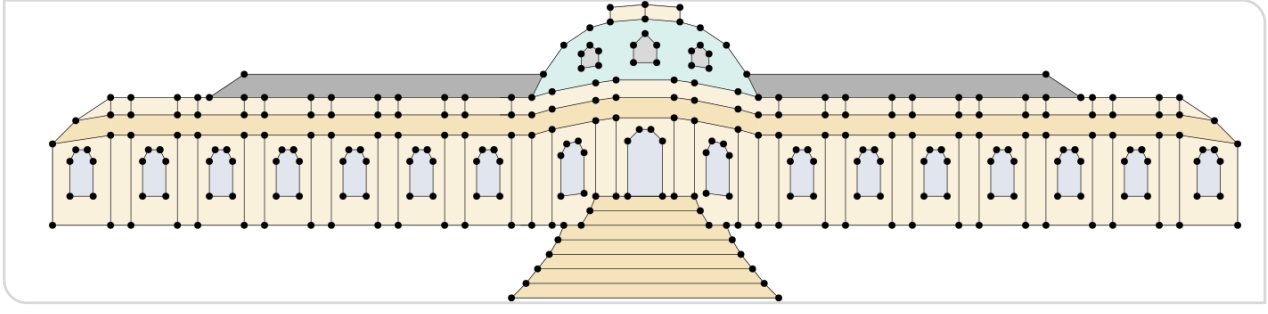
Karlsruhe Institute of Technology

Efficient Recognition of Subgraphs of Planar Cubic Bridgeless Graphs

Miriam Goetze, Paul Jungeblut, Torsten Ueckerdt | 07.09.2022



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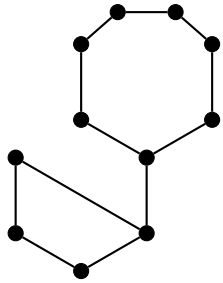

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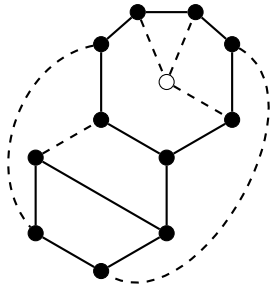

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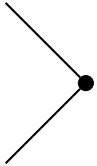

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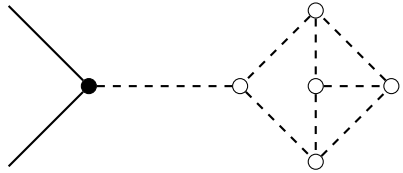

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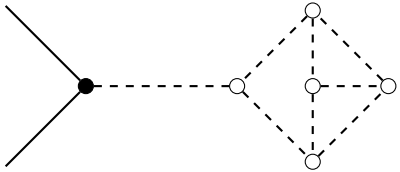

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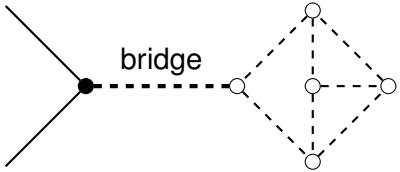

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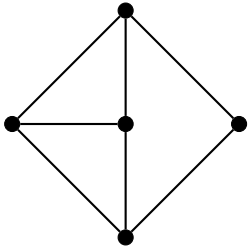

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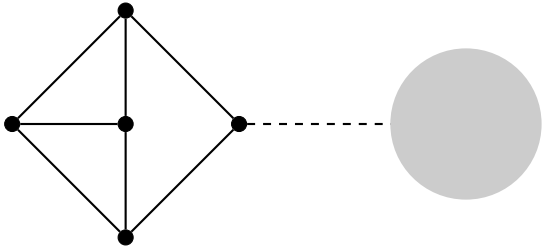

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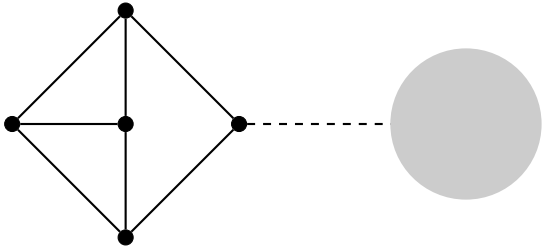

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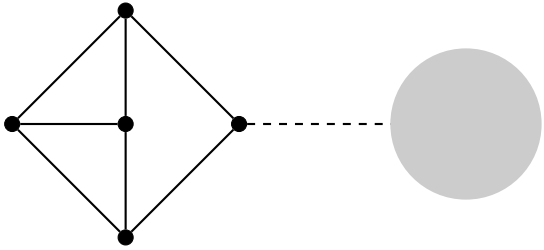

sometimes impossible

Theorem (Tait)

G planar, cubic, bridgeless
$\Longrightarrow$ 3-edge-colorable

(Follows from the 4-Color-Theorem.)

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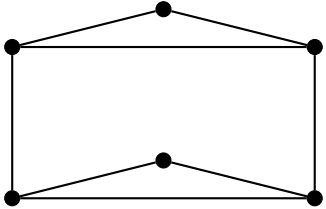
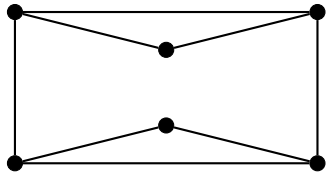


Variable Embedding



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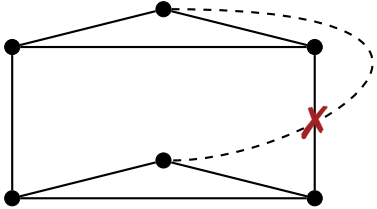

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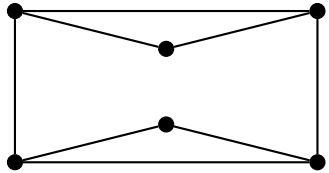

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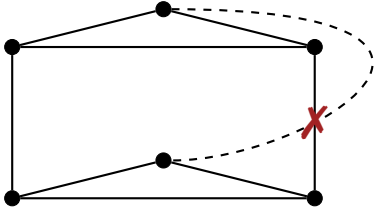

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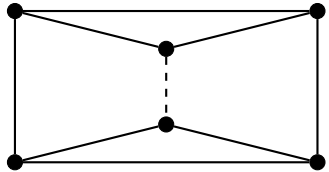

Our Results

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Theorem

3-AUGMENTATION can be solved in $\mathcal{O}(n^2)$ time.

- for fixed and variable embedding
- construct supergraph H if it exists

Discussion

Efficient recognition of a large subclass of 3-edge-colorable graphs:

- 3-EDGE-COLORING is NP-complete (planarity not required)
- open for planar graphs

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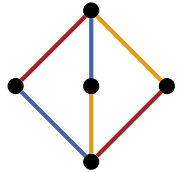
Sadly:



3-edge-colorable,
but $\nexists$ 3-augmentation

Proof Idea

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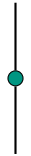

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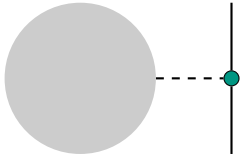

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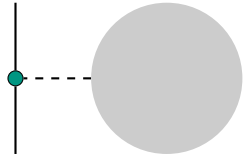

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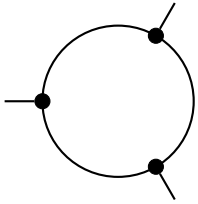

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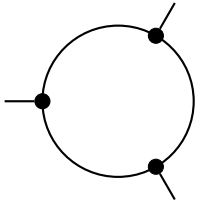

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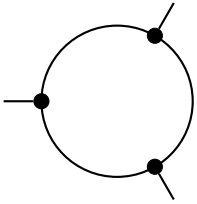

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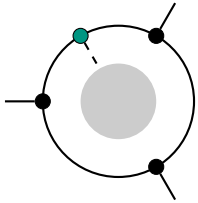

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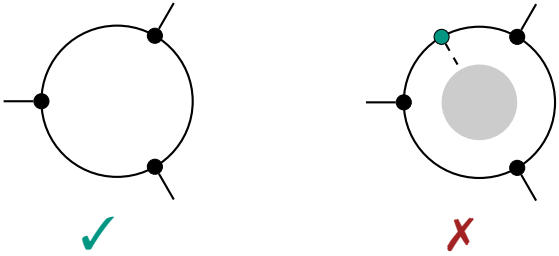

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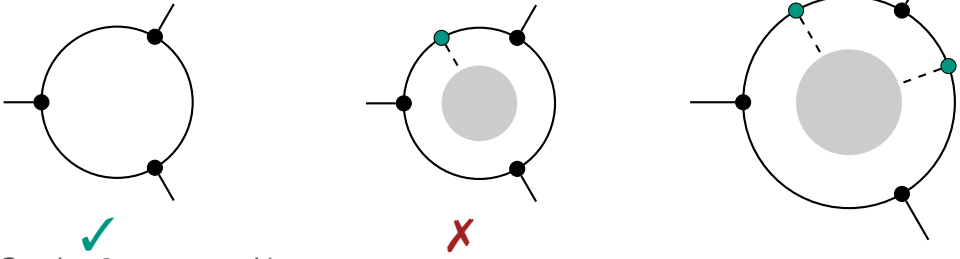

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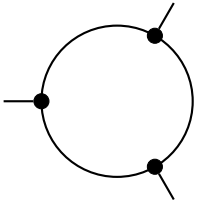

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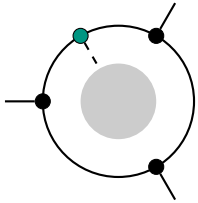

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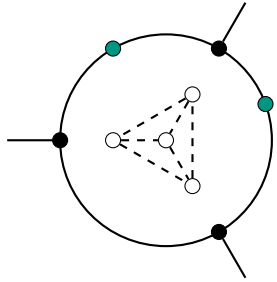

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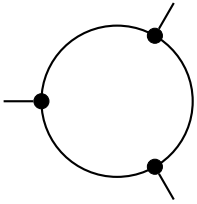

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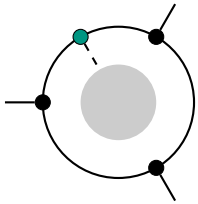

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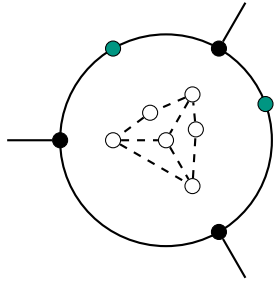

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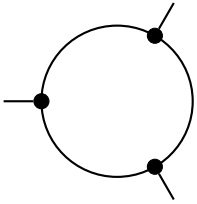

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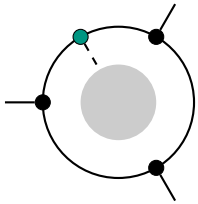

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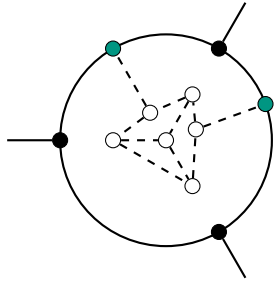

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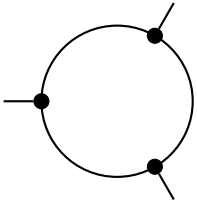

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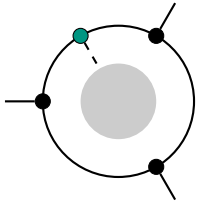

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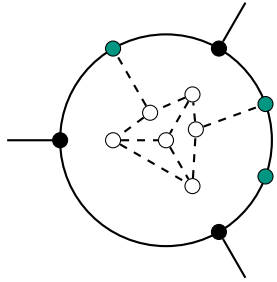

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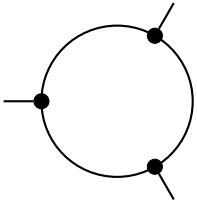

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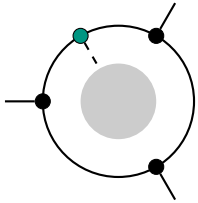

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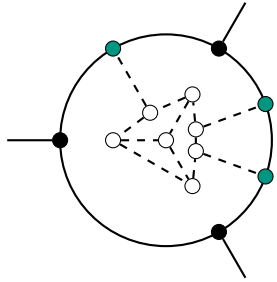

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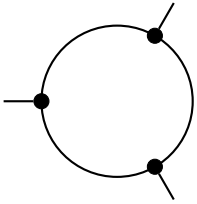

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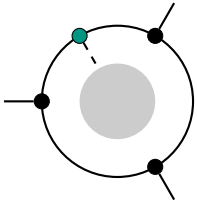

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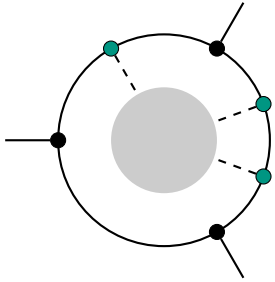

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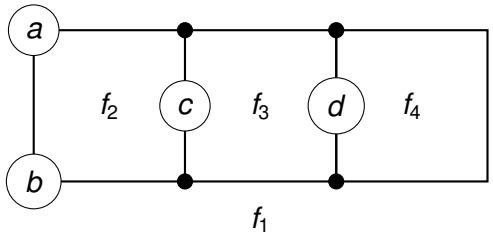
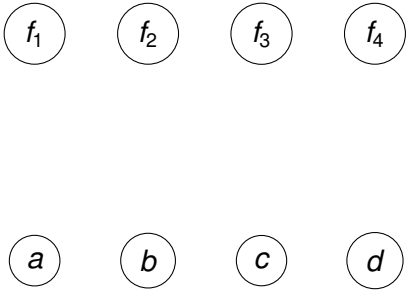


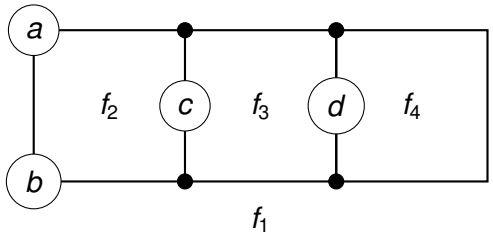


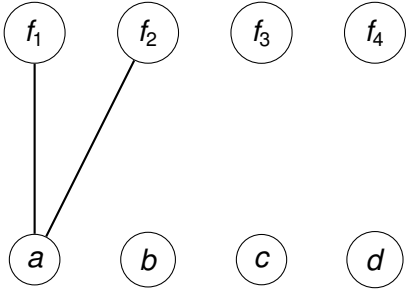
(Assume G to be 2-connected.)

Proof Idea

Problem: fixed 3-AUGMENTATION

- **Input**: **embedded** planar graph G with $\Delta(G) \leq 3$
- **Question**: $\exists$ planar, cubic, bridgeless supergraph H?





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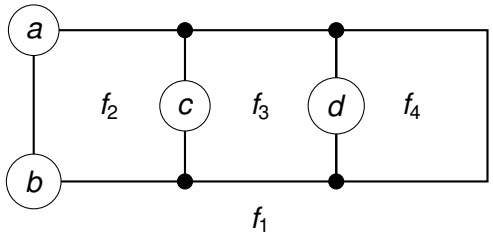

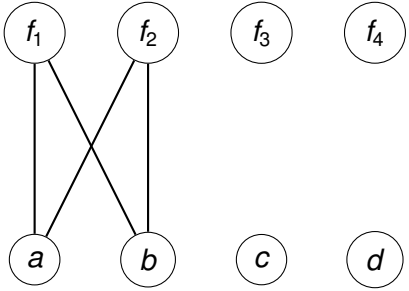

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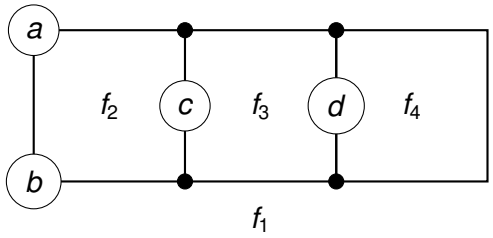

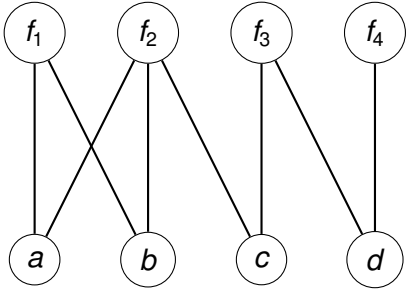

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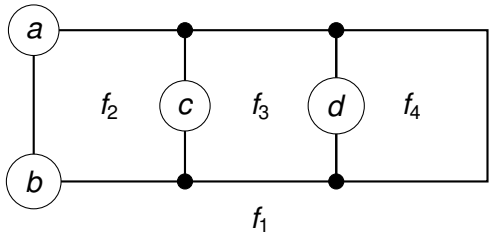

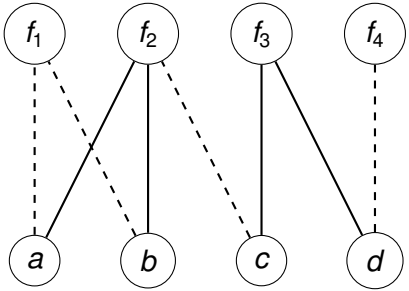

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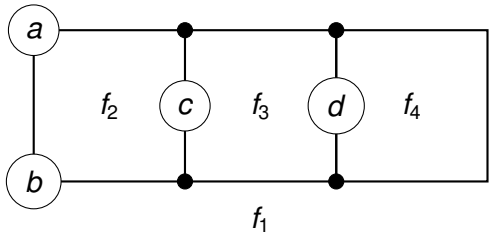

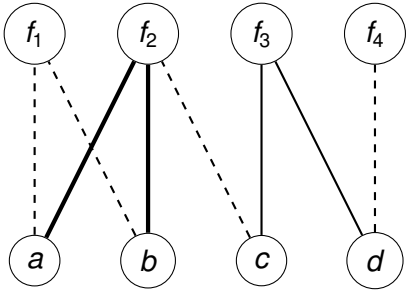

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Proof Idea

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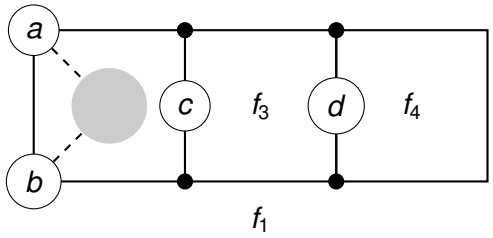

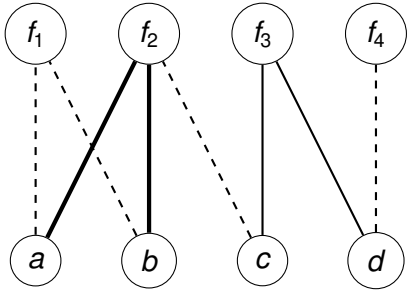

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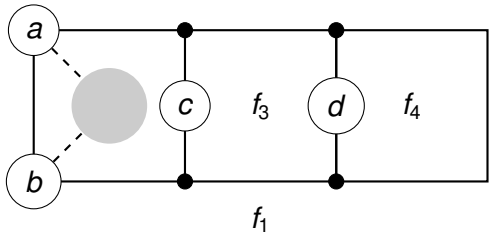

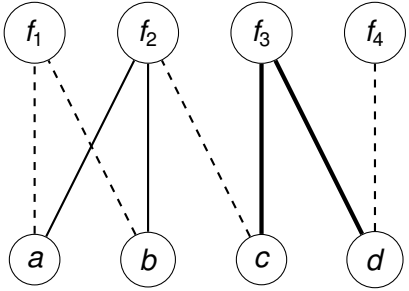

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Proof Idea

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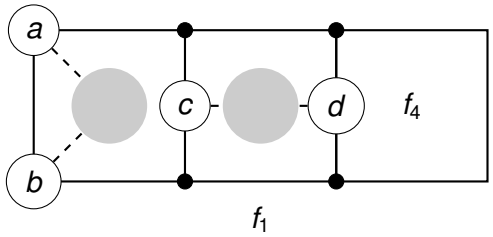
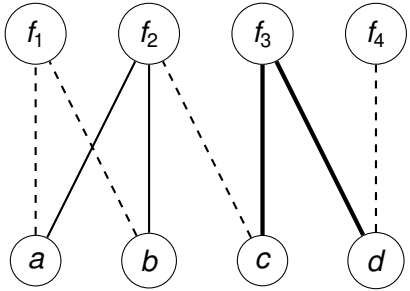




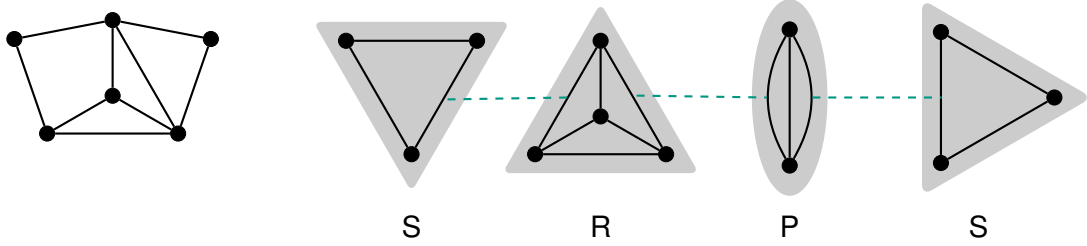
(Assume G to be 2-connected.)

Proof Idea

Problem: variable 3-AUGMENTATION

- **Input**: planar graph G with $\Delta(G) \leq 3$
- **Question**: $\exists$ planar, cubic, bridgeless supergraph H?

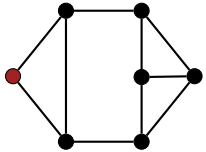
Dynammic Programming on SPQR-Tree (representation of all embeddings)



Open Problems

PLANAR 3-EDGE-COLORING

- Grötzsch' Conjecture

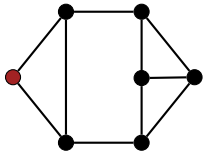


only obstruction

Open Problems

PLANAR 3-EDGE-COLORING

- Grötzsch' Conjecture



only obstruction

Generalization of 3-AUGMENTATION

- 4,5-AUGMENTATION
  (fixed ✓, variable **?**)
- 2-connected 4,5-AUGMENTATION
  (fixed ✓, variable **?**)