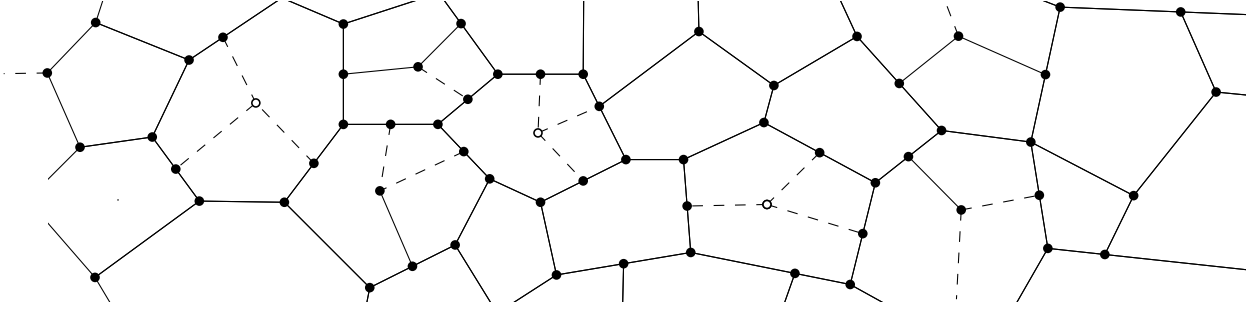
Karlsruhe Institute of Technology

Recognition Complexity of Subgraphs of k-Connected Planar Cubic Graphs

Miriam Goetze, Paul Jungeblut, Torsten Ueckerdt | 12.04.2024



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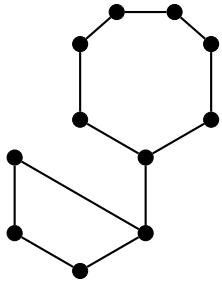

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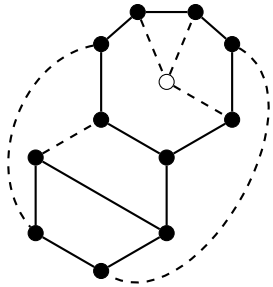

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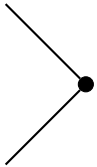

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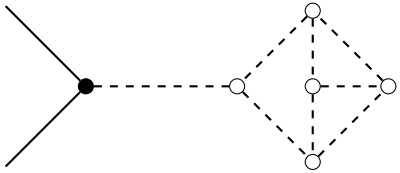

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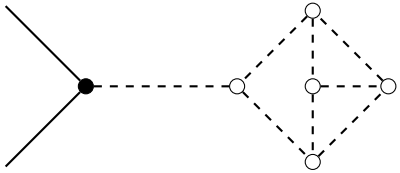

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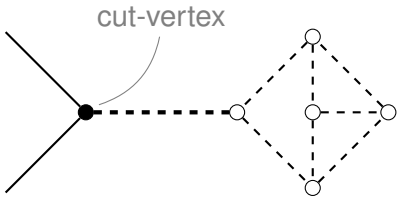

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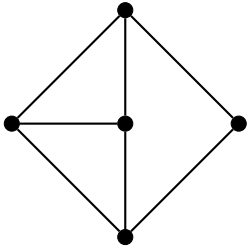

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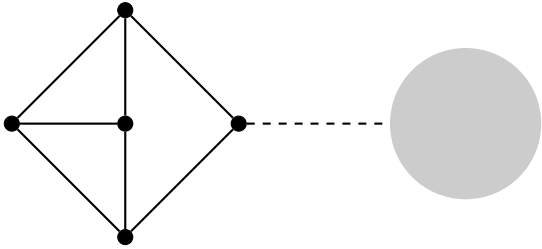

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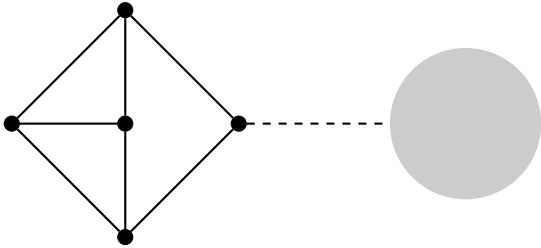

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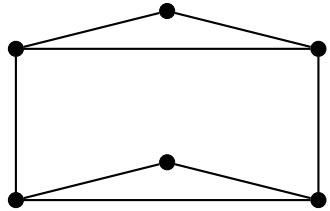
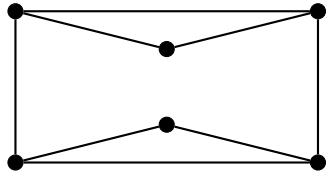


Variable Embedding



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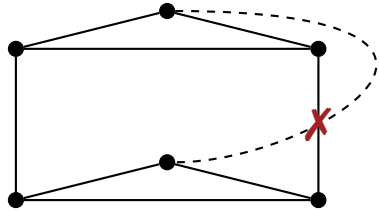

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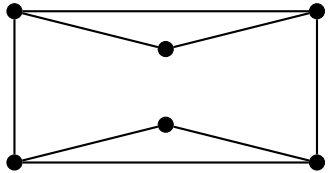

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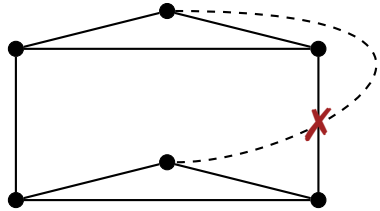

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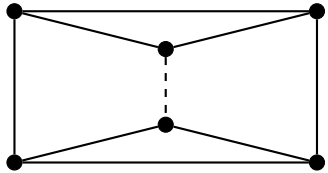

Our Results

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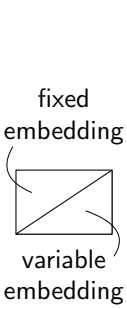

input G connectivity \ output H connectivity	any	con.	2-con.	3-con.
any	✓ / ✓	P / P	P / P	? / NPC
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ESA 2022

EuroCG 2024

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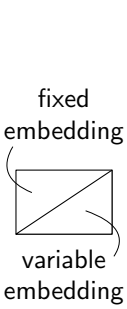

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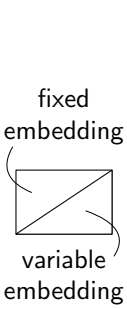

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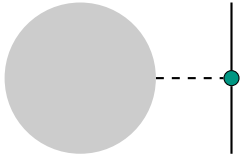

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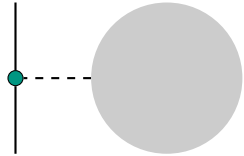

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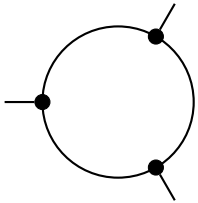

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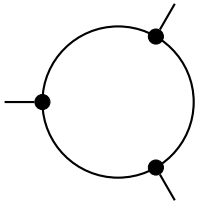

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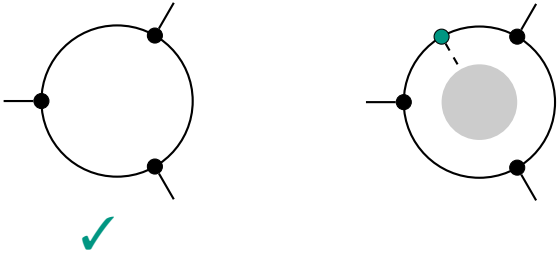

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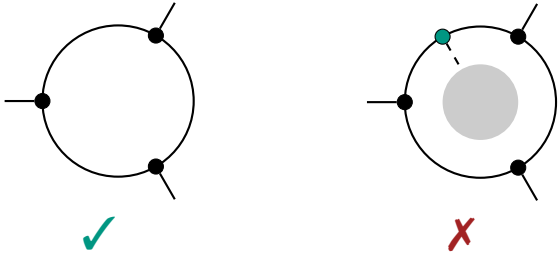

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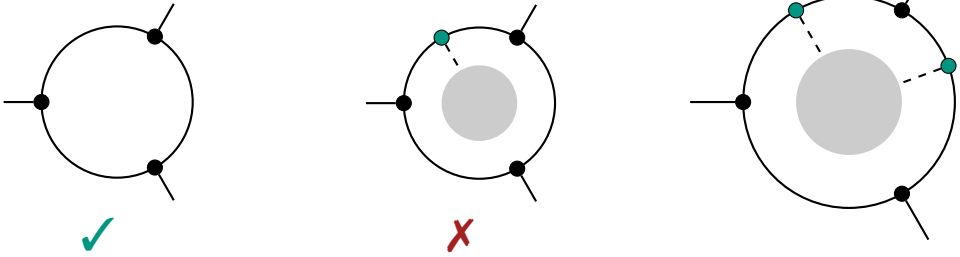

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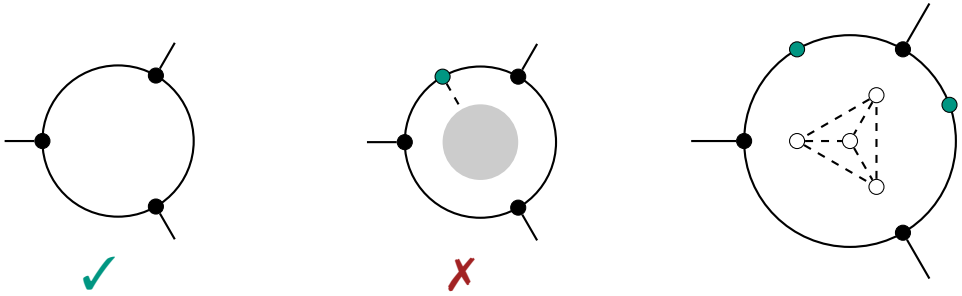

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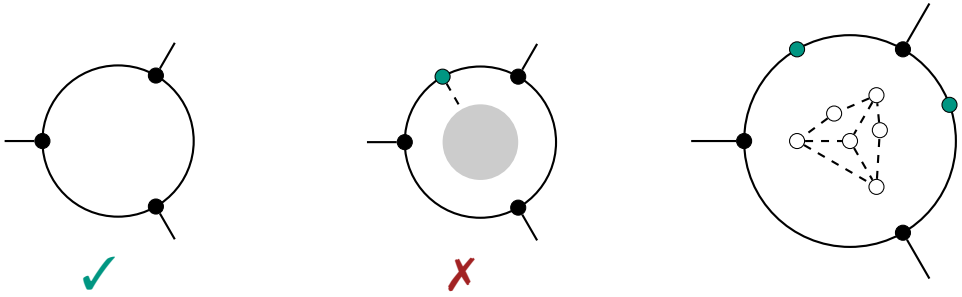

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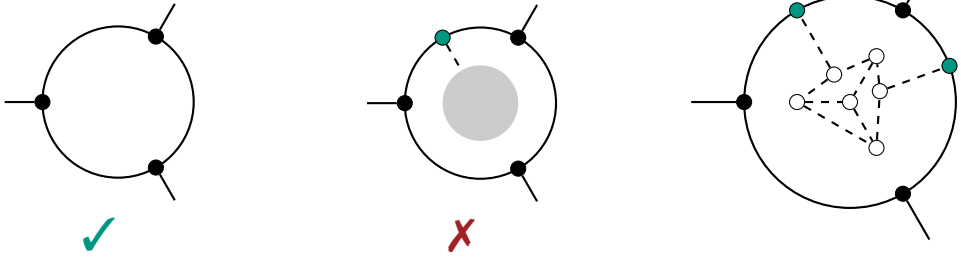

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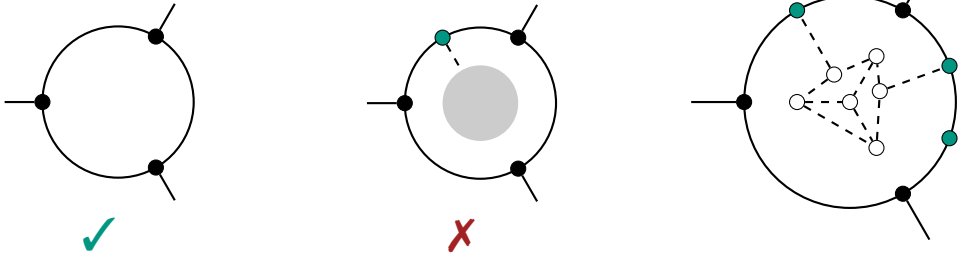

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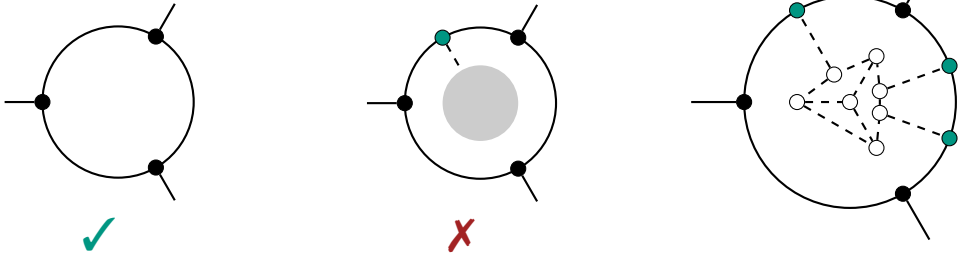

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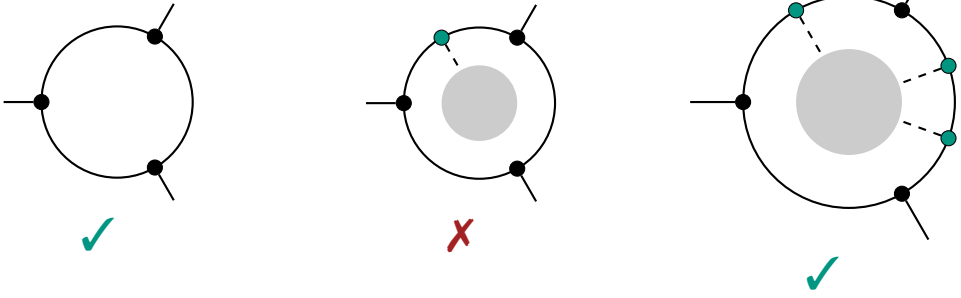

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Generalized Factors

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- **Input**: graph G and $\forall v \in V(G)$ a set $B_v \subseteq \mathbb{N}_0$.
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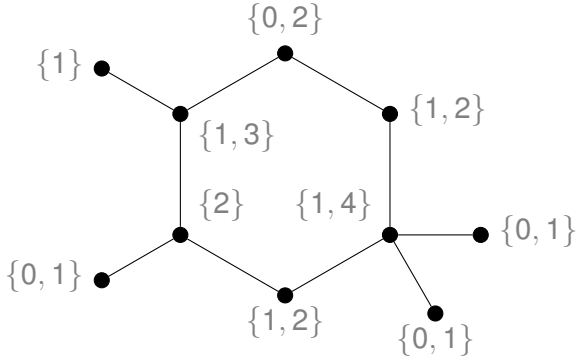

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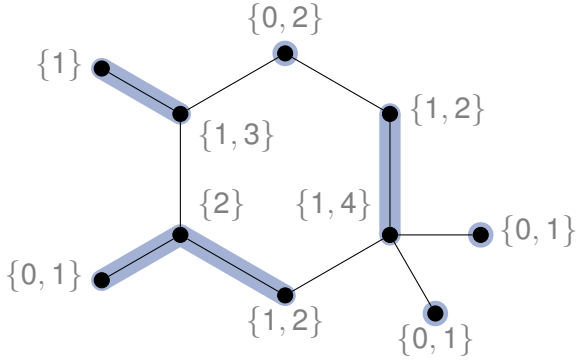

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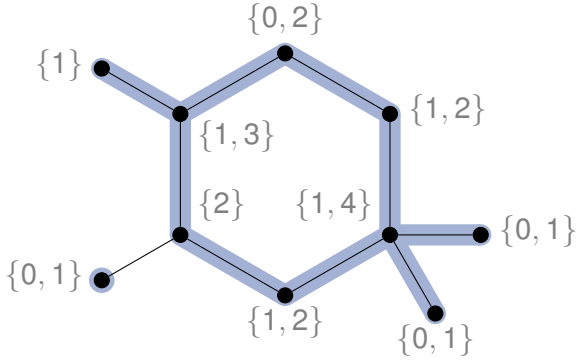

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Definition: Gaps

B_v has a gap of length ℓ if $\exists i$:

- $i \in B_v$
- $i+1, i+2, \ldots, i+\ell \notin B_v$
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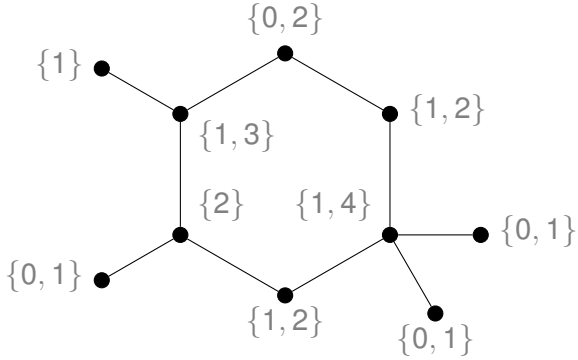

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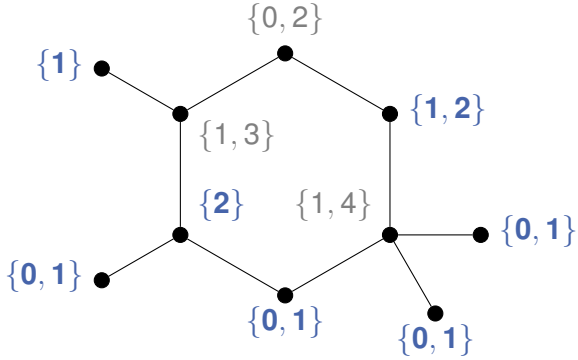
no gaps



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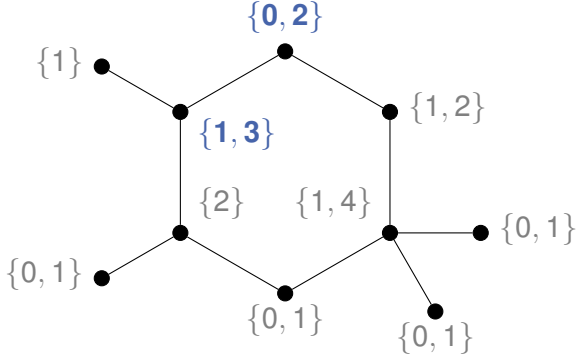
gaps of length 1



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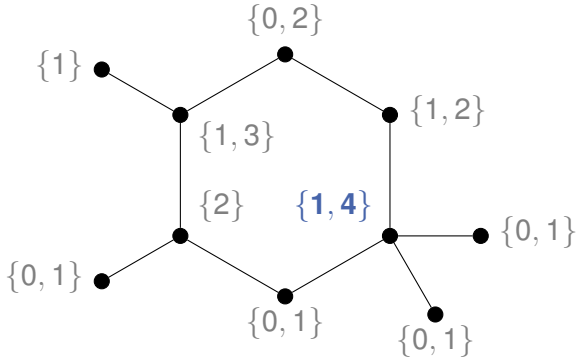
gaps of length 2



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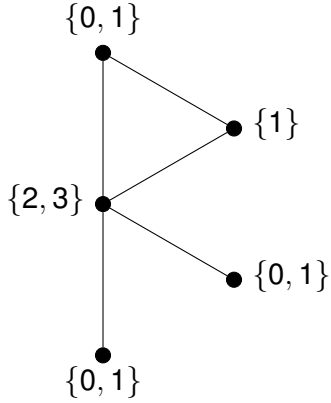
Theorem (Lovàsz, 1972)

GeneralizedFactor is $\mathcal{NP}$-complete.



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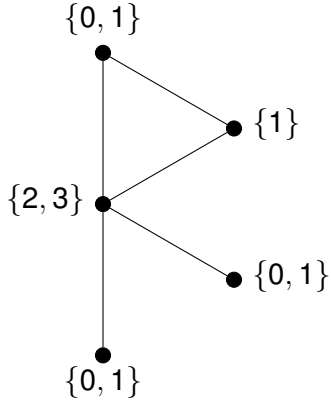
- **Question**: $\exists$ spanning $H \subseteq G$ with $\forall v\colon\ \deg_H(v) \in B_v$?

Theorem (Lovàsz, 1972)

GeneralizedFactor is $\mathcal{NP}$-complete.

Theorem (Cornuéjols, 1988)

If there are no gaps of length ≥ 2, then a GeneralizedFactor can be computed in $\mathcal{O}(|V(G)|^4)$.



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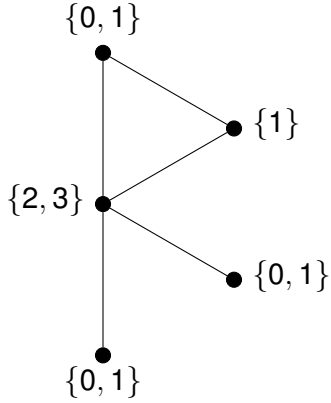

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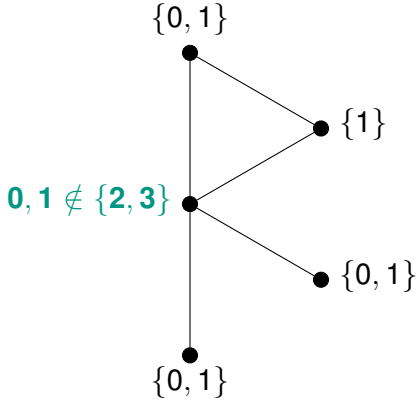

Proof Idea

Problem: fixed 2-connected 3-AUGMENTATION

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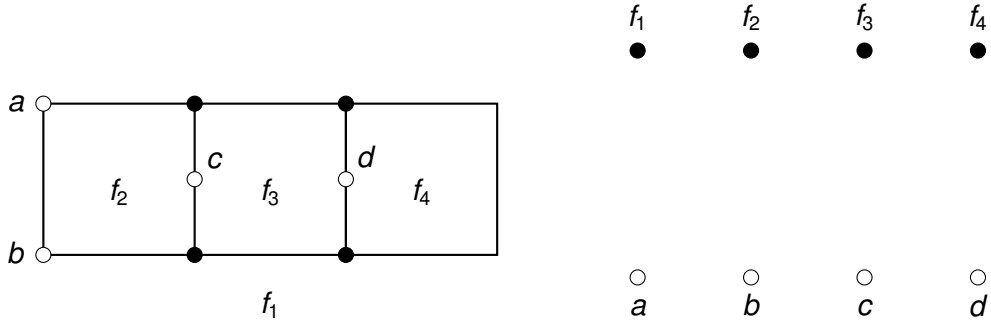

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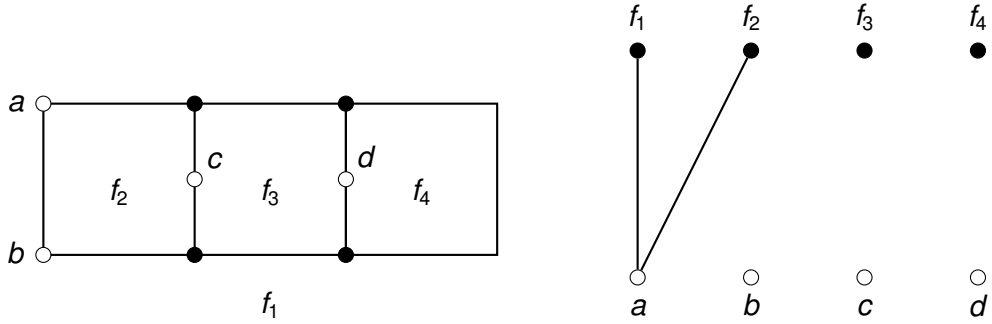

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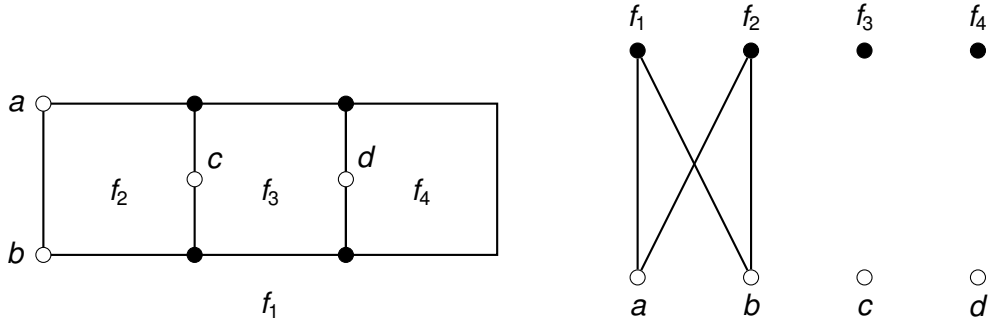

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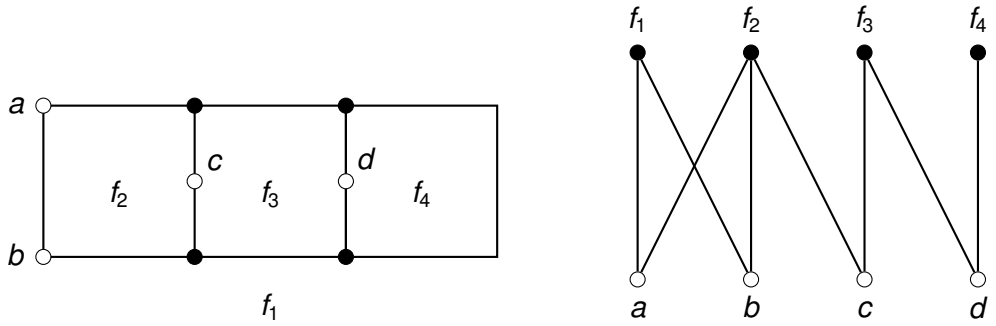

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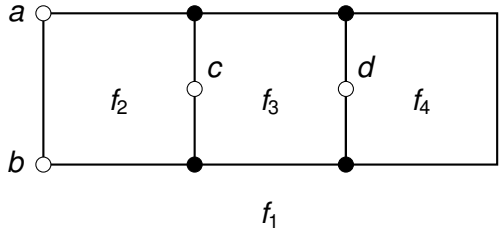

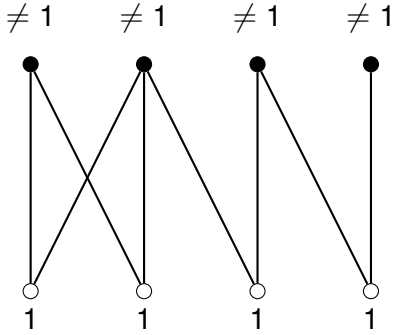

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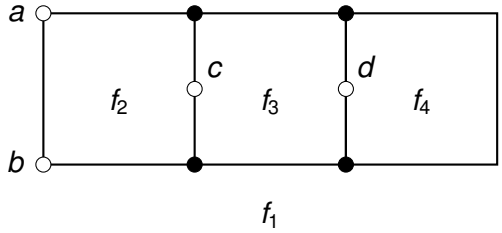

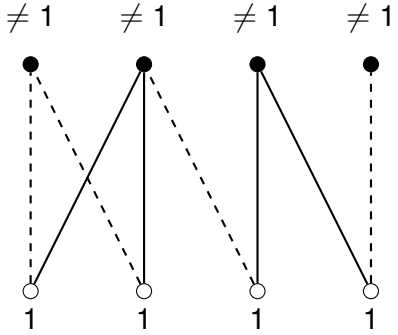

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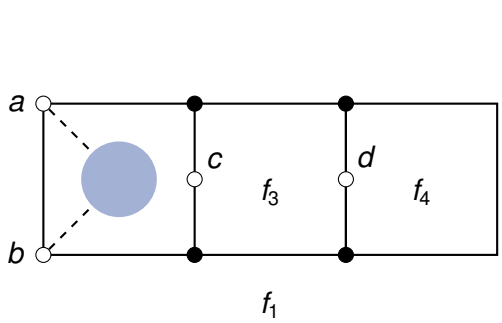

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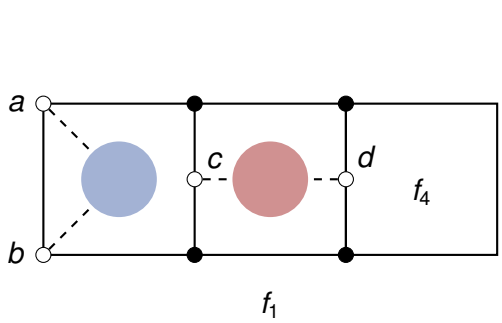

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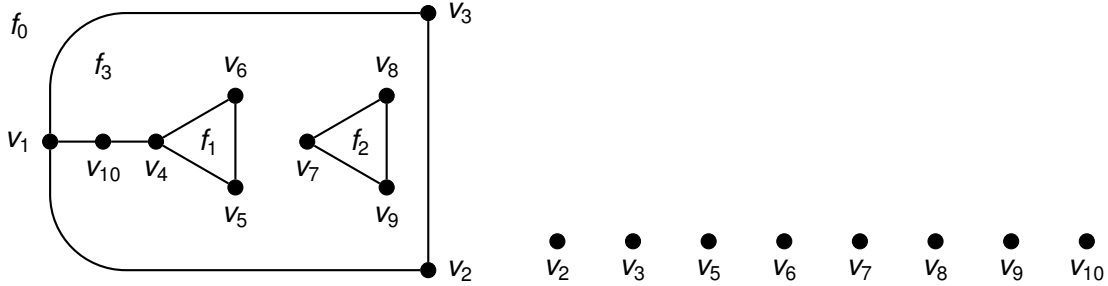

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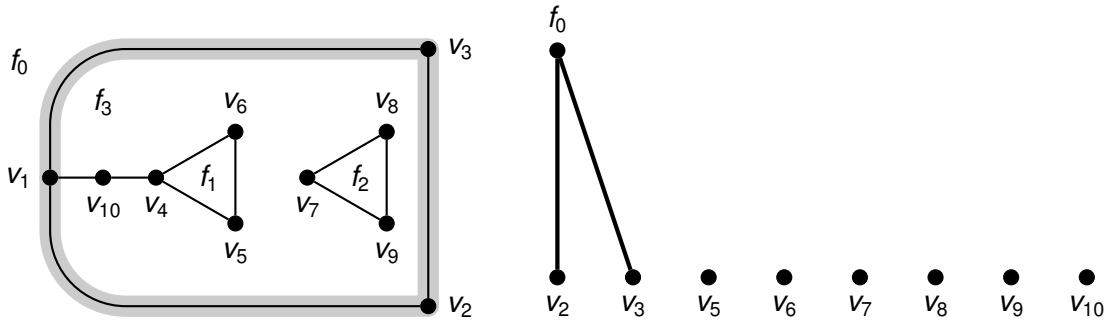

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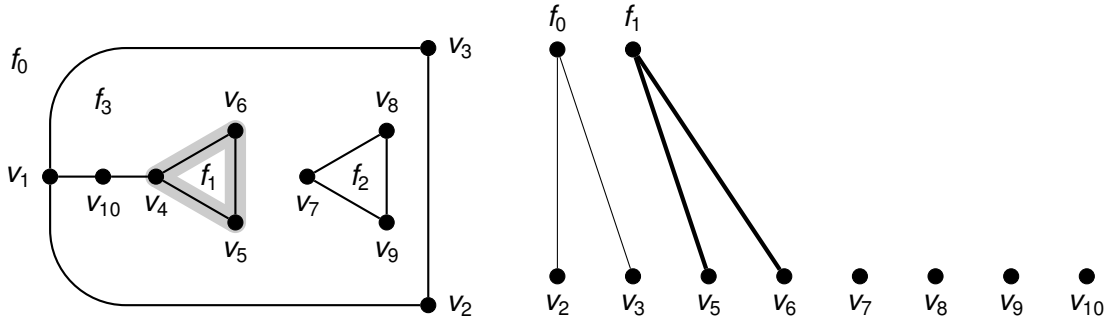

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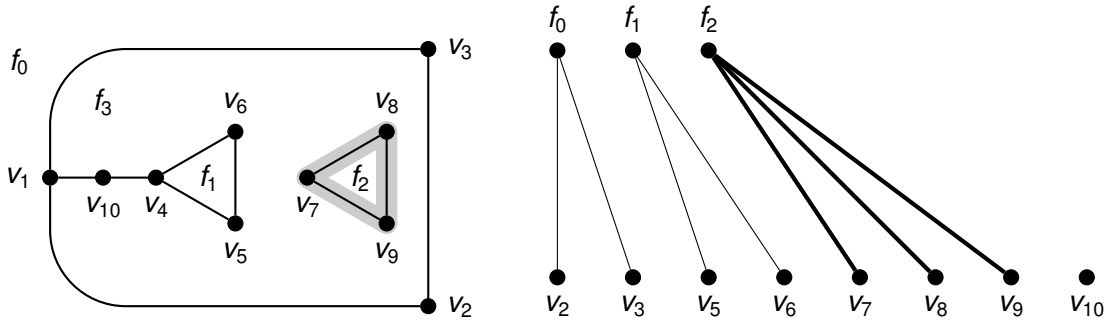

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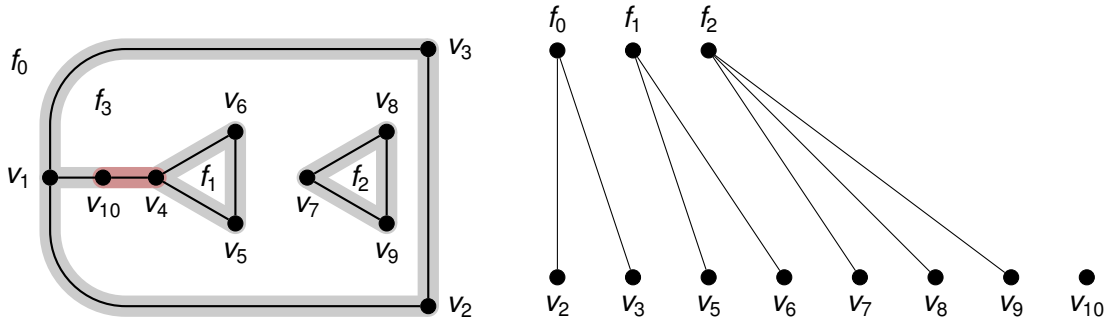

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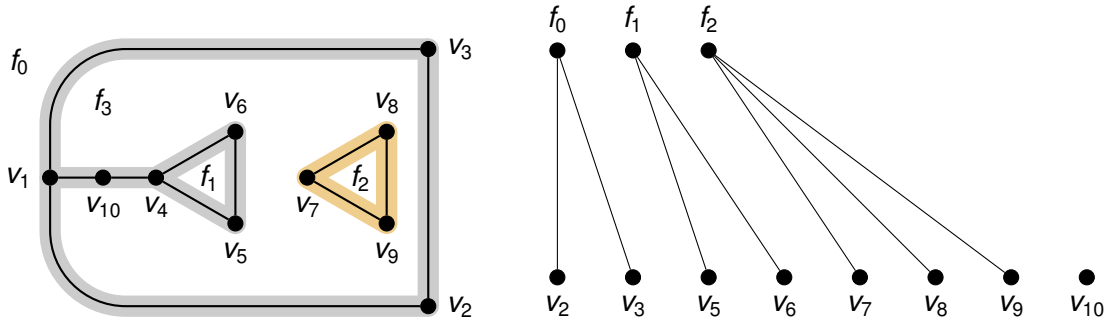

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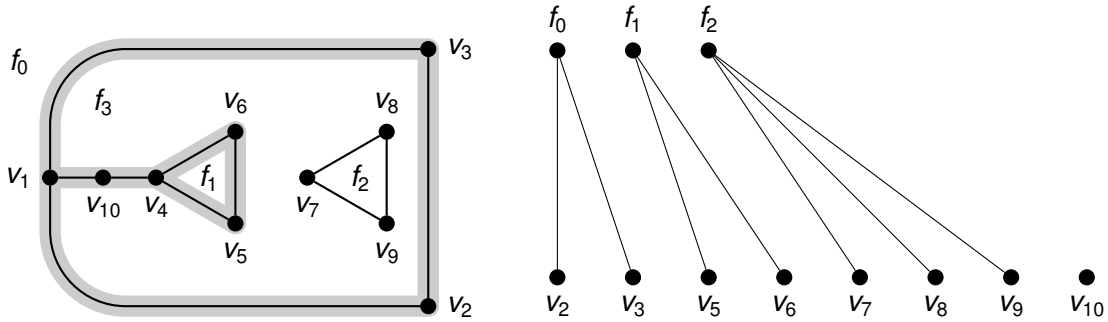

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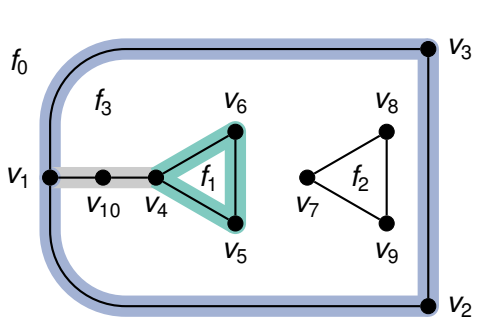

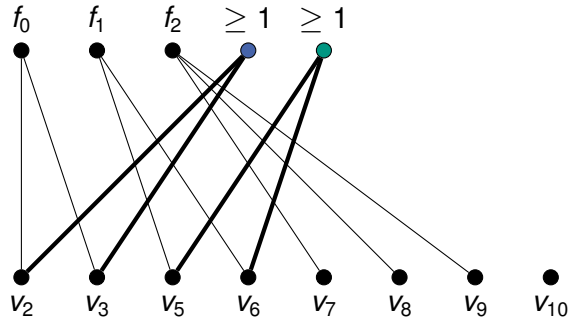

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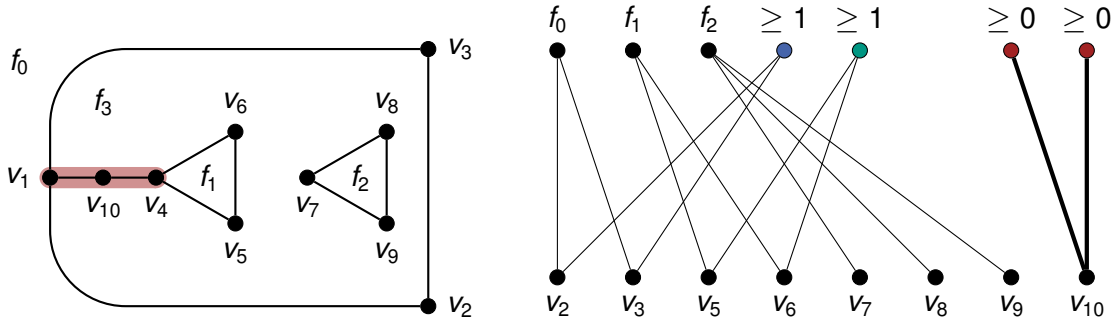

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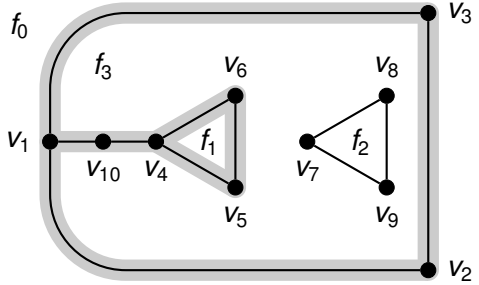

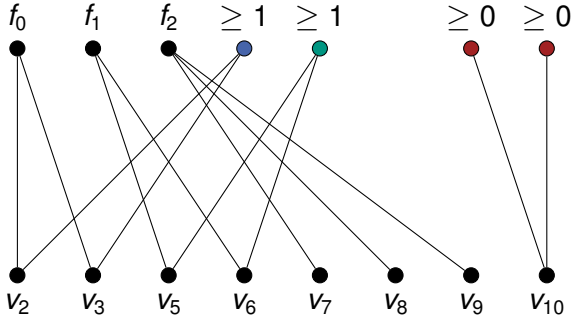

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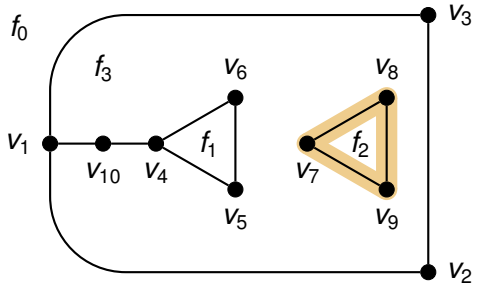

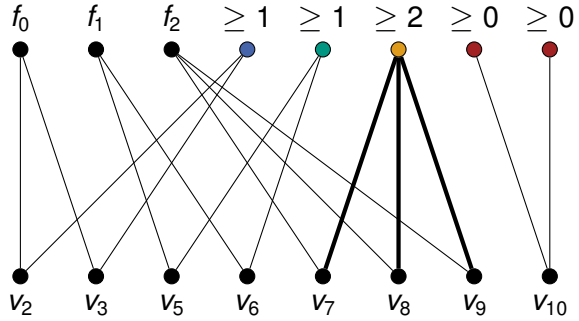

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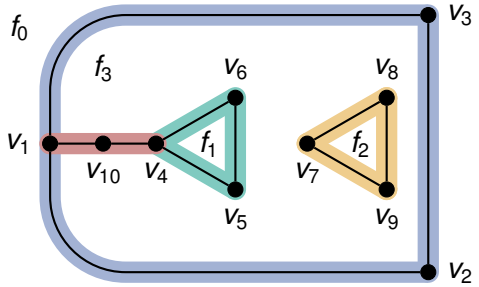
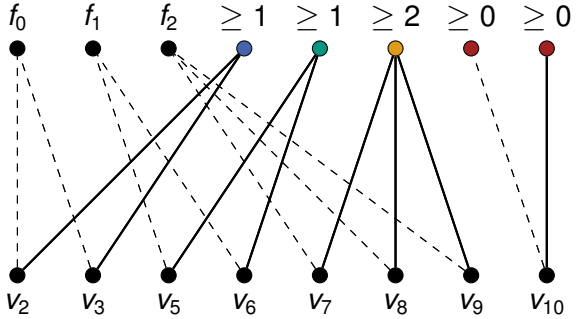
Cornuéjols $\mathcal{O}(|V(G)|^4)$





NP-Completeness

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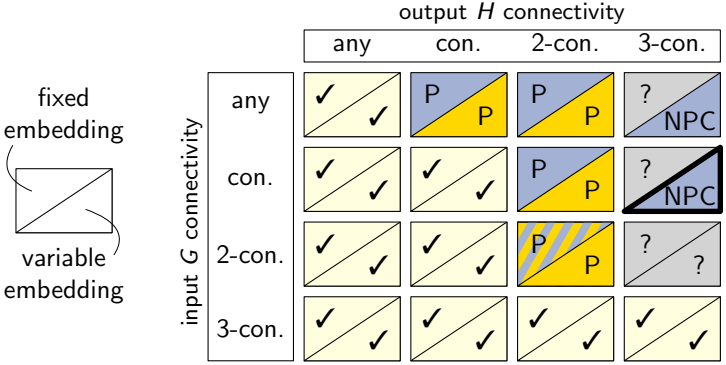
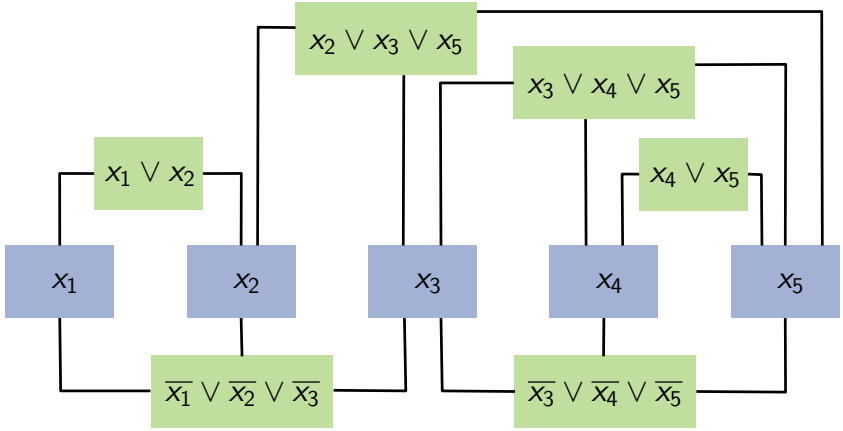


ESA 2022

EuroCG 2024

NP-Completeness

Problem: PLANARMONOTONE-3-SAT



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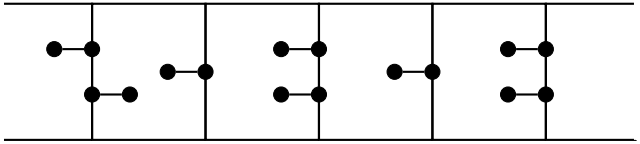

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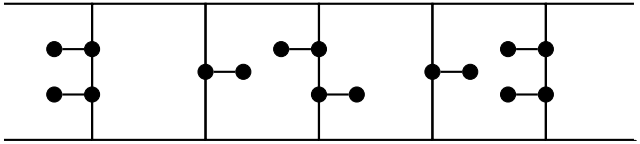

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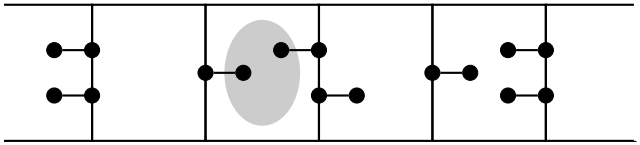

✗

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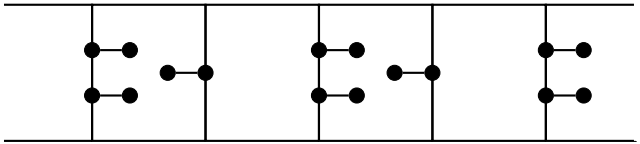

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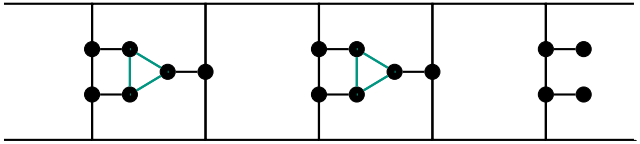

✓

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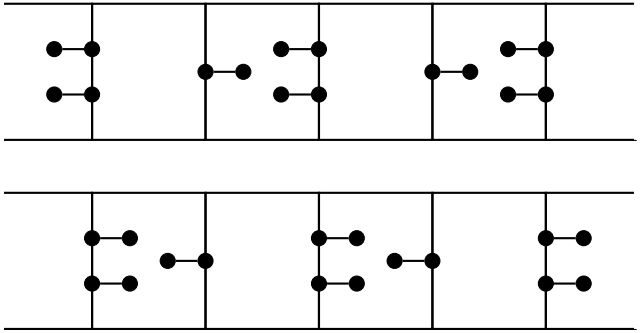

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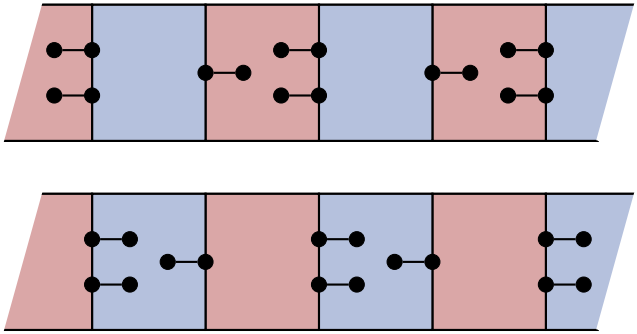

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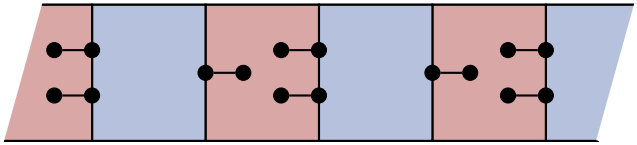
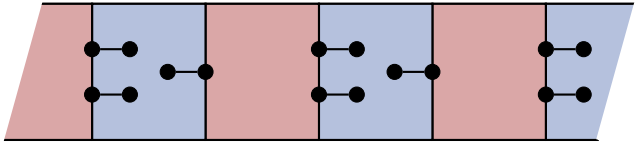


false configuration



true configuration

NP-Completeness Clause Gadget

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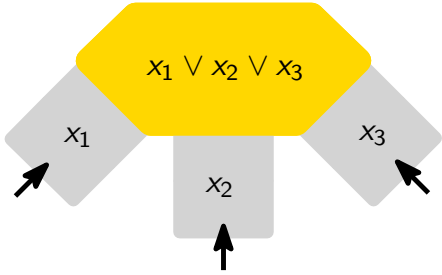

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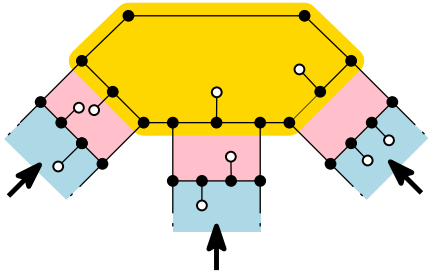

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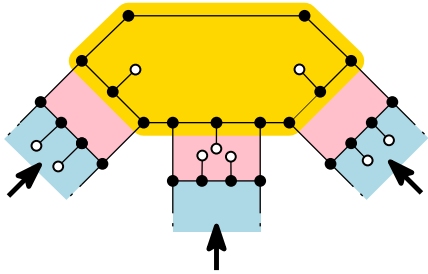

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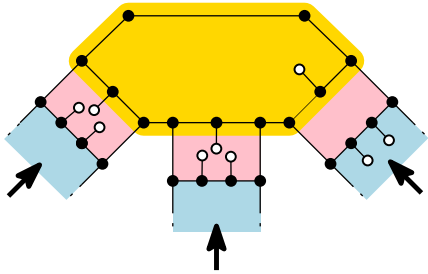

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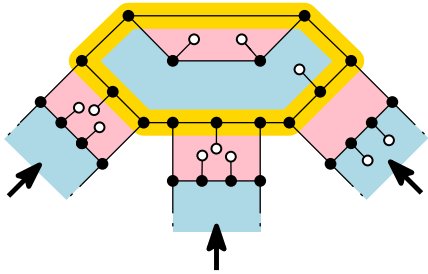
- **Question**: $\exists$ planar cubic **3-connected** supergraph H?



NP-Completeness Clause Gadget

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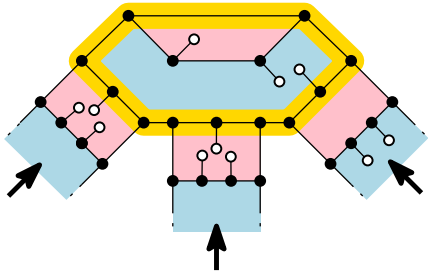

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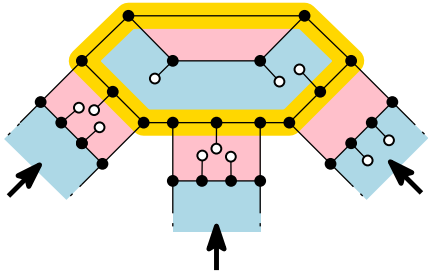

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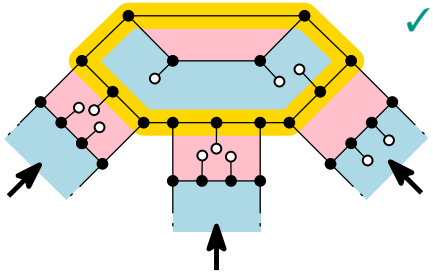

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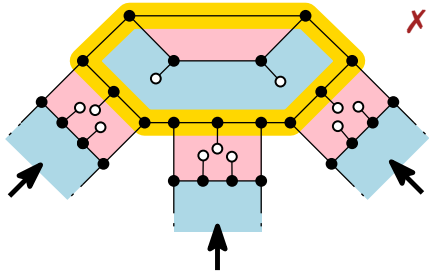

NP-Completeness

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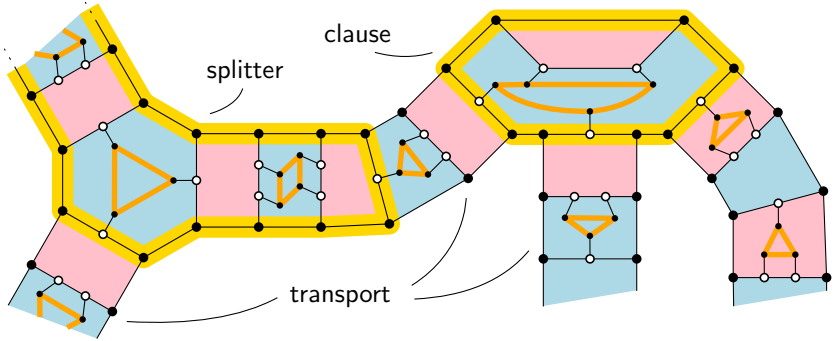
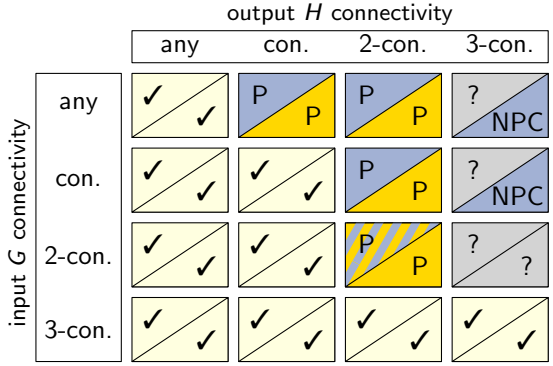


Open Questions

Problem: 3-AUGMENTATION



Problem: r-AUGMENTATION

- **Input**: planar graph G with $\Delta(G) \leq r$
- **Question**: $\exists$ planar r-regular supergraph H?

Values: $r \in \{4, 5\}$