A Sublinear Bound on the Page Number of Upward Planar Graphs

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upward planar graph
upward planar graph

definition:
height: max # of pairwise comparable vertices

in the diagram, the upward planar graph is depicted with arrows indicating the direction of the edges.
upward planar graph

height: max # of pairwise comparable vertices
width: max # of pairwise incomparable vertices
An upward planar graph

- **height**: max # of pairwise comparable vertices
- **width**: max # of pairwise incomparable vertices

Find a topological vertex ordering and partition the edges into pages such that there is no monochromatic 1-vertex path.

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Page Number

\[ \text{Page Number} \] \[pn(G) = \min k \text{ such that there is } k\text{-page book embedding for } G\]
Open Problem

Is the page number of upward planar graphs bounded?

Open for:
- upward planar graphs
- planar posets
- planar lattices
- upward outerplanar graphs
- upward planar 2-trees

Bounded for:
- single-source upward outerplanar graphs (Bhore et al. 2021)
- upward outerpaths (Nöllenburg, Pupyrev 2021)
- upward planar 3-trees (Frati et al. 2013)
- upward planar graphs whose 4-connected components have bounded page number (Frati et al. 2013)
Lower Bound

There is an upward planar graph $G$ with $\text{pn}(G) \geq 4$ (Hung 1993).

**Theorem**

There is an upward planar graph $G$ with $\text{pn}(G) \geq 5$. 
Upper Bounds for Upward Planar Graphs

Frati et al. (2013): $O(h \log n)$

Theorem

$\text{pn}(G) \leq O(h \log h)$
for $h = \text{height}(G)$

Theorem

$\text{pn}(G) \leq 14 \cdot \text{width}(G)$

Trivial: $O(n)$

Theorem

The page number of $n$-vertex upward planar graphs is $O(n^{2/3} \log^{2/3}(n))$. 
Height
= \# \text{ vertices in a longest path}

Theorem
pn(G) \leq O(h \log h) \text{ for } h = \text{height}(G)
Width

= max # pairwise incomparable vertices

partition vertices into width(G) paths
Width

= max # pairwise incomparable vertices

partition vertices into width(G) paths
Width

= max # pairwise incomparable vertices

partition vertices into
width(\(G\)) paths

Theorem

14 \cdot width(\(G\)) pages suffice with any topological vertex ordering
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small width + small height = small page number
Repeatedly

- choose a long path and
- apply width-theorem to subgraph of small width
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- apply width-theorem to subgraph of small width

For any topological ordering $O(n^{2/3} \log^{2/3}(n))$ pages suffice
Repeatedly
- choose a long path and
- apply width-theorem to subgraph of small width

Remaining subgraph
- no long paths
- apply height-theorem to subgraph of small height

For any topological ordering\( O(n^{2/3} \log^{2/3}(n)) \) pages suffice

Construct ordering s.t. \( O(n^{2/3} \log^{2/3}(n)) \) pages suffice for remaining edges
Page number of upward planar graphs

- lower bound: 5
- upper bound: $O(n^{2/3} \log^{2/3}(n))$
Page number of upward planar graphs

- lower bound: 5
- upper bound: $O(n^{2/3} \log^{2/3}(n))$

What is the maximum page number of
- upward planar graphs,
- planar posets,
- upward / acyclic outerplanar graphs?