Karlsruhe Institute of Technology

A Sublinear Bound on the Page Number of Upward Planar Graphs

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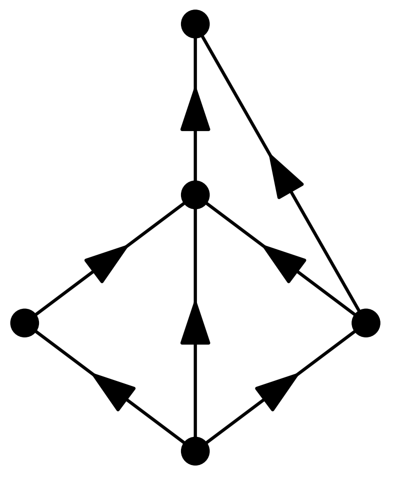
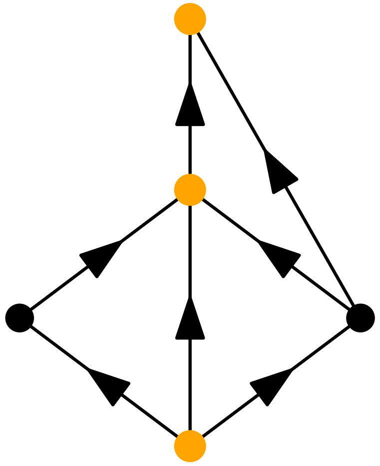
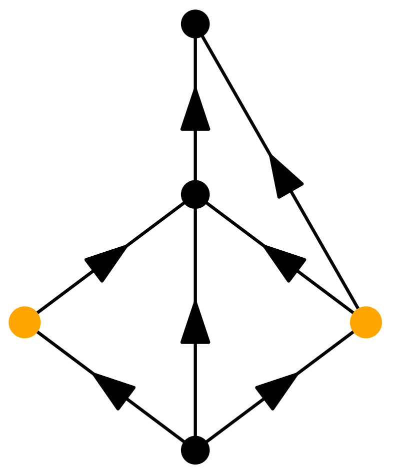


upward planar graph



upward planar graph

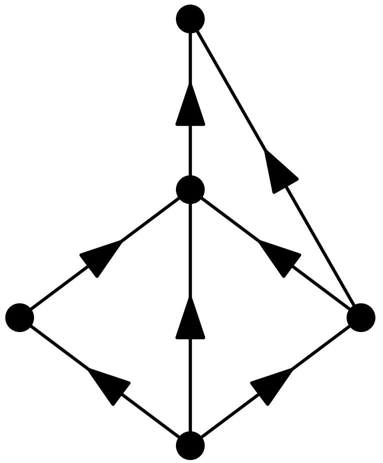
height: max # of pairwise comparable vertices



upward planar graph

height: max # of pairwise comparable vertices

width: max # of pairwise incomparable vertices



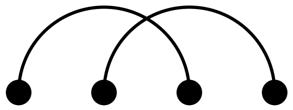
upward planar graph

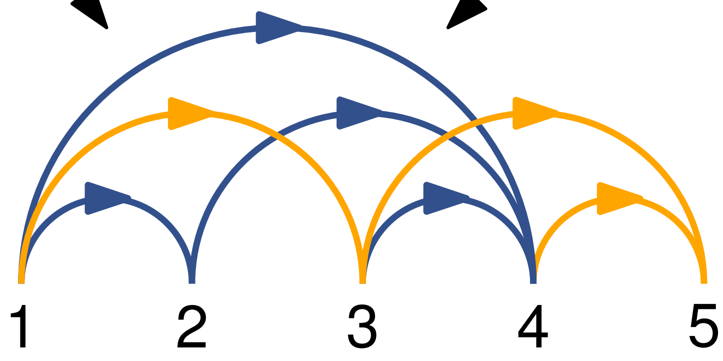
height: max # of pairwise comparable vertices

width: max # of pairwise incomparable vertices

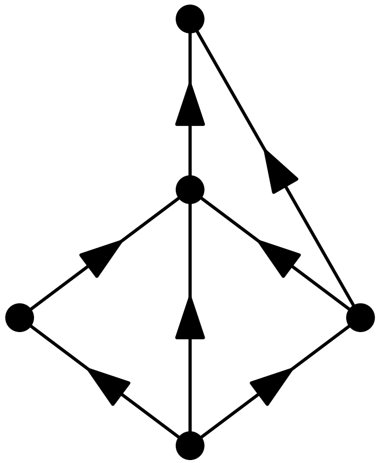
find a partition of
the edges into pages

no monochromatic





find topological vertex ordering



upward planar graph

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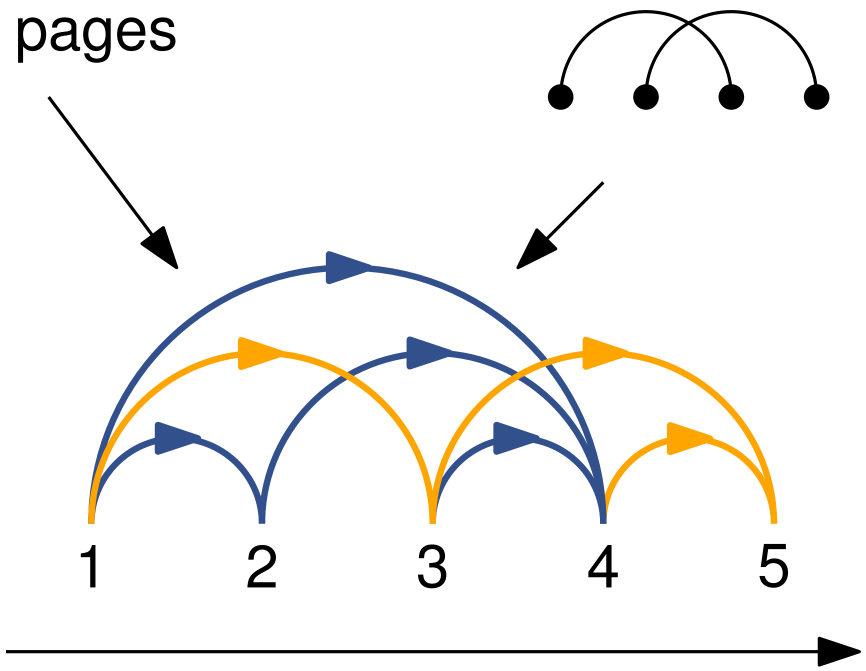


find topological vertex ordering

height: max # of pairwise comparable vertices
width: max # of pairwise incomparable vertices

Page Number

pn(G) = min k such that there is k-page book embedding for G

Open Problem

Is the page number of upward planar graphs bounded?

Open for:

- upward planar graphs
- planar posets
- planar lattices
- upward outerplanar graphs
- upward planar 2-trees

Bounded for:

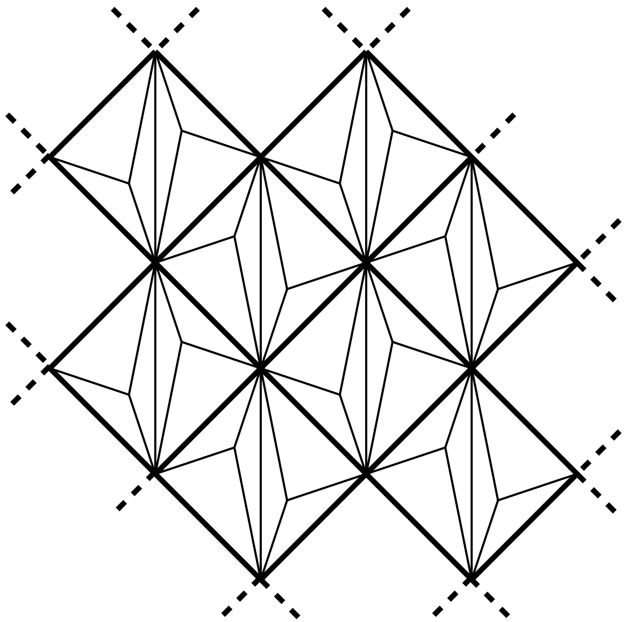
- single-source upward outerplanar graphs (Bhore et al. 2021)
- upward outerpaths (Nöllenburg, Pupyrev 2021)
- upward planar 3-trees (Frati et al. 2013)
- upward planar graphs whose 4-connected components have bounded page number (Frati et al. 2013)

Lower Bound

There is an upward planar graph G with $\mathrm{pn}(G) \geqslant 4$ (Hung 1993).

Theorem

There is an upward planar graph G with $\mathrm{pn}(G) \geqslant 5$.



Upper Bounds for Upward Planar Graphs

Frati et al. (2013): $O(h \log n)$

Theorem

$\mathrm{pn}(G) \leqslant O(h \log h)$
for $h = \mathrm{height}(G)$

Theorem

$\mathrm{pn}(G) \leqslant 14 \cdot \mathrm{width}(G)$

Trivial: $O(n)$

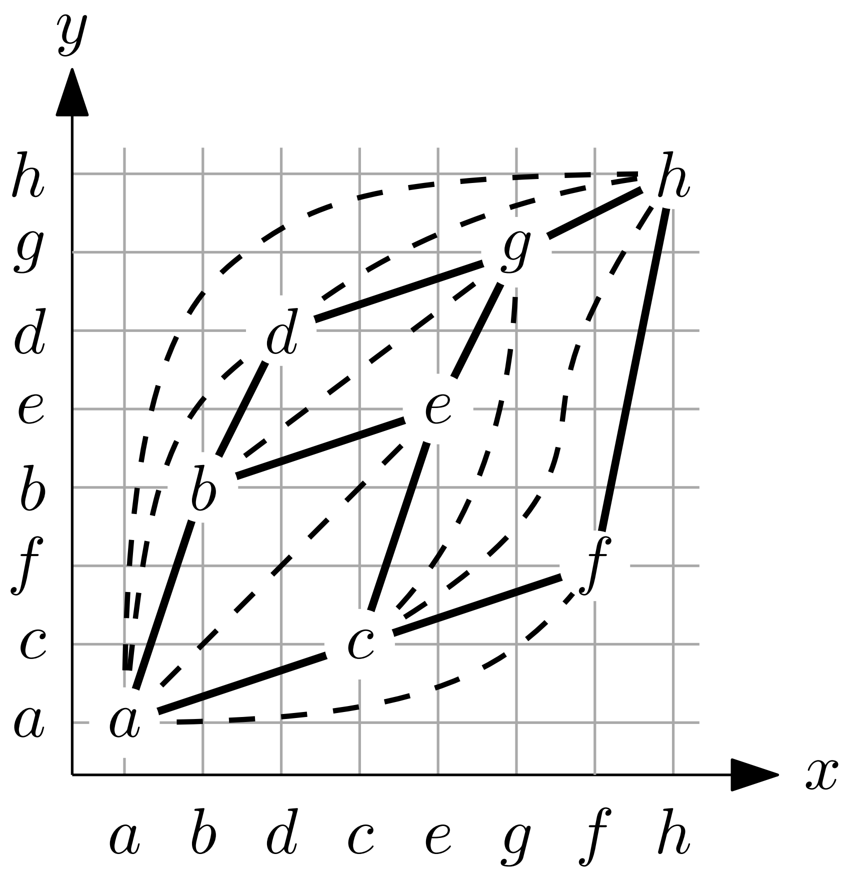
Theorem

The page number of n-vertex upward planar graphs is $O(n^{2/3} \log^{2/3}(n))$.

Height

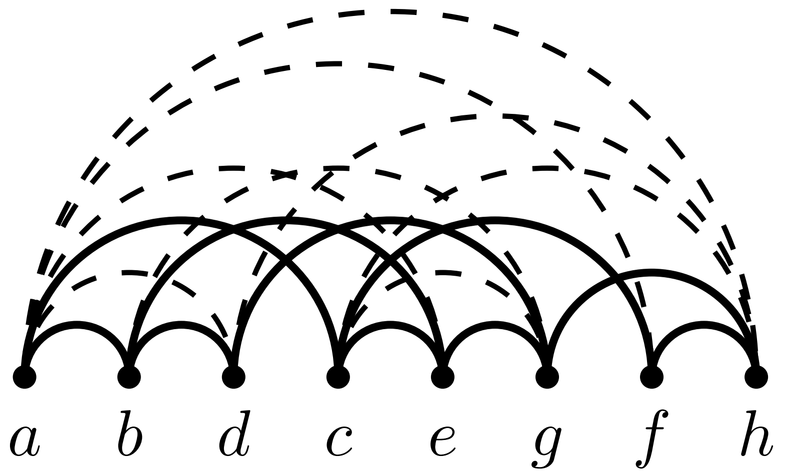
= # vertices in a longest path

dominance drawing



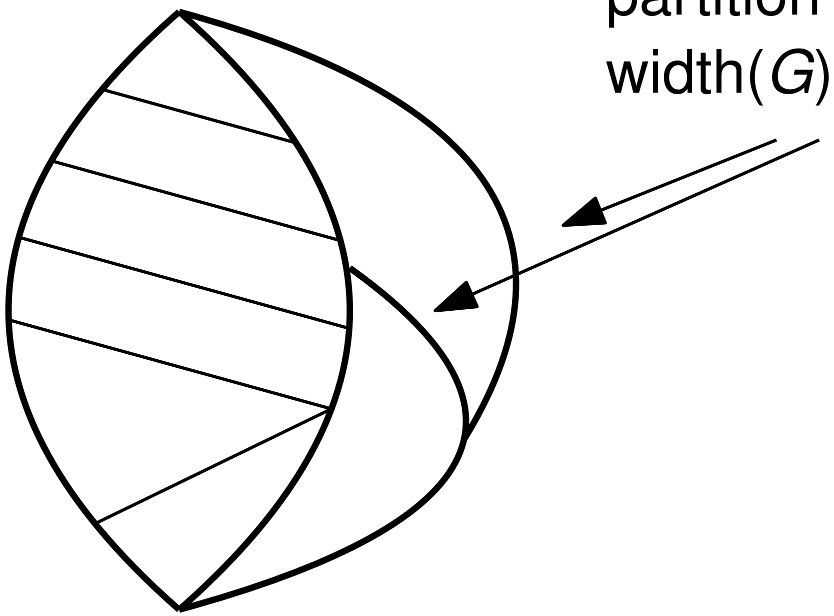
Theorem

$\mathrm{pn}(G) \leqslant O(h \log h)$ for $h = \mathrm{height}(G)$



Width

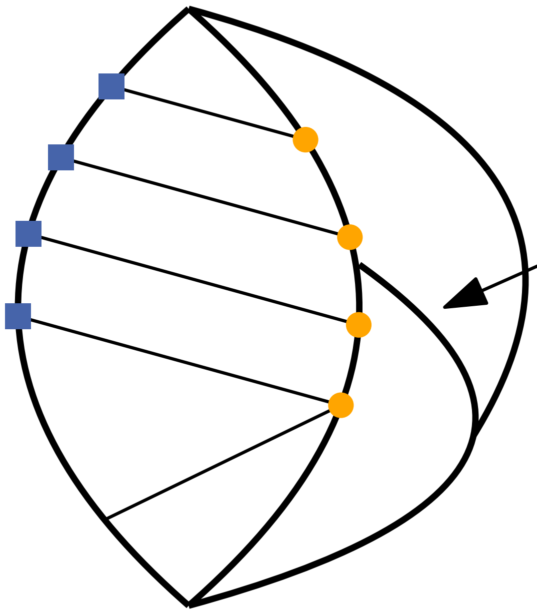
= max # pairwise incomparable vertices



partition vertices into width(G) paths

Width

= max # pairwise incomparable vertices



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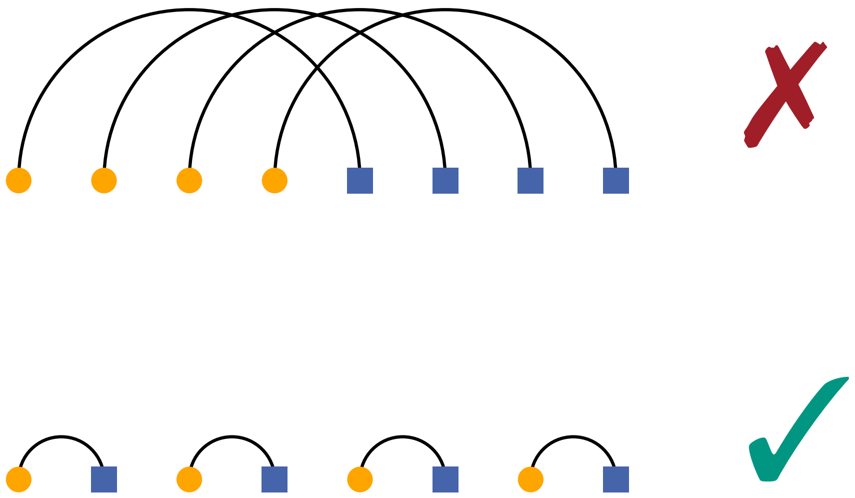

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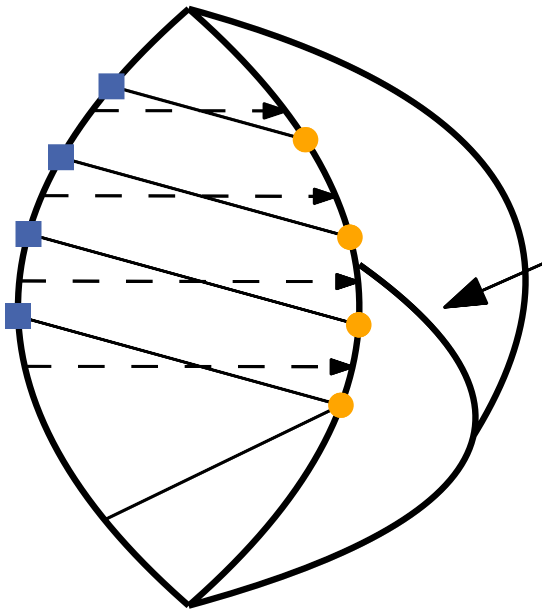

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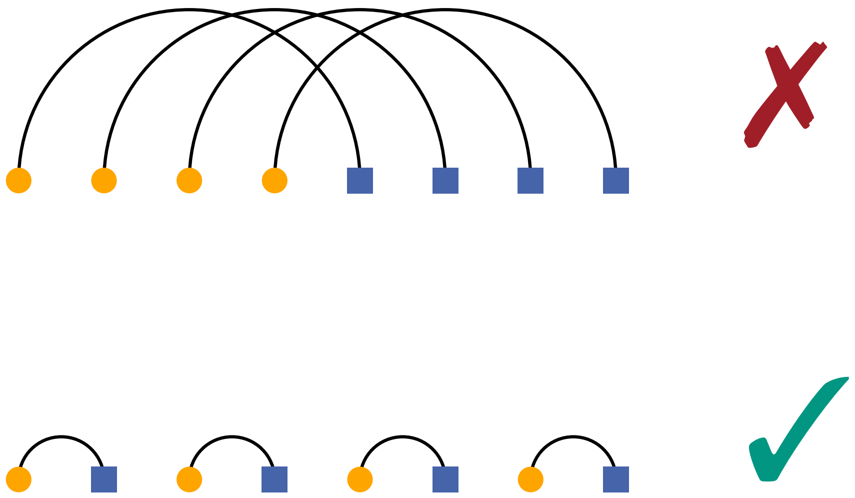
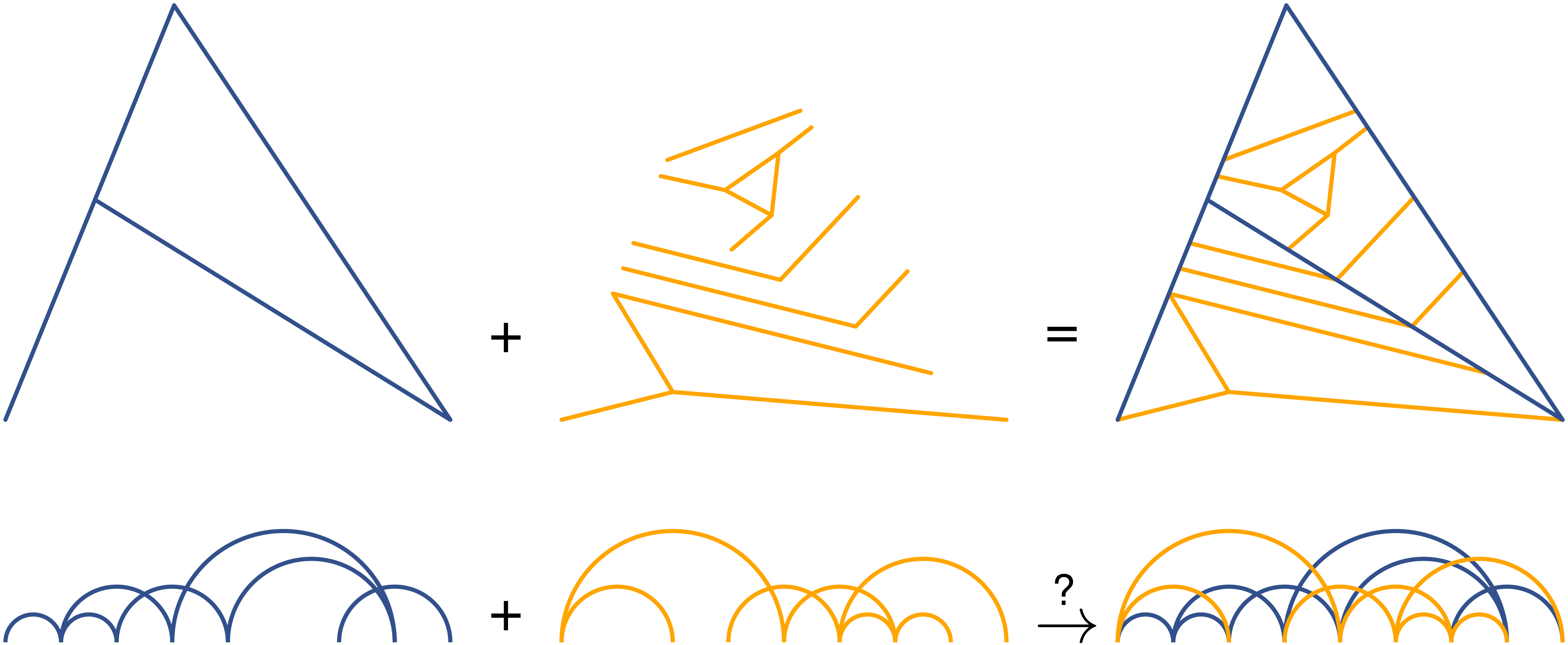


Theorem

$14 \cdot \text{width}(G)$ pages suffice with any topological vertex ordering



small width small height small page number

Repeatedly

- choose a long path and
- apply width-theorem to subgraph of small width

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- choose a long path and
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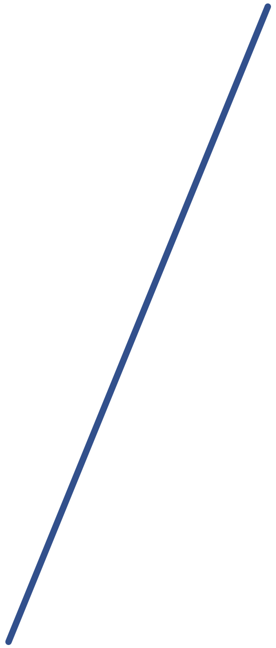

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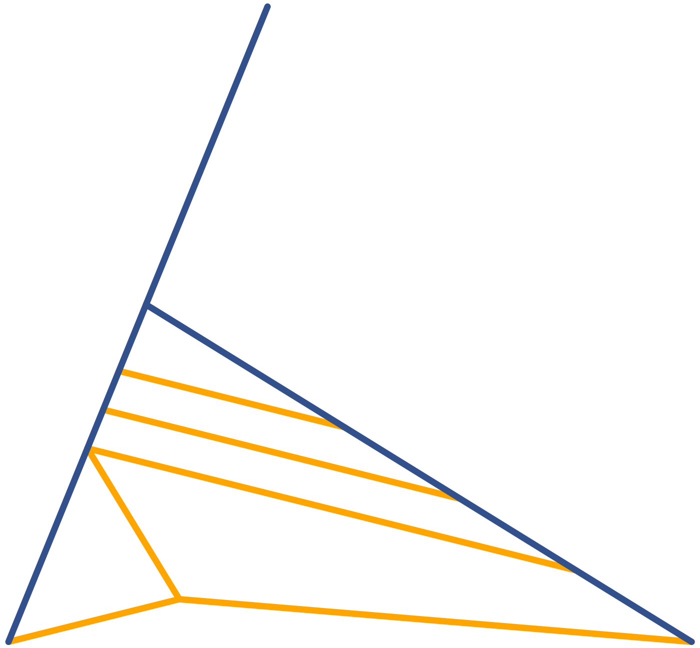

Repeatedly

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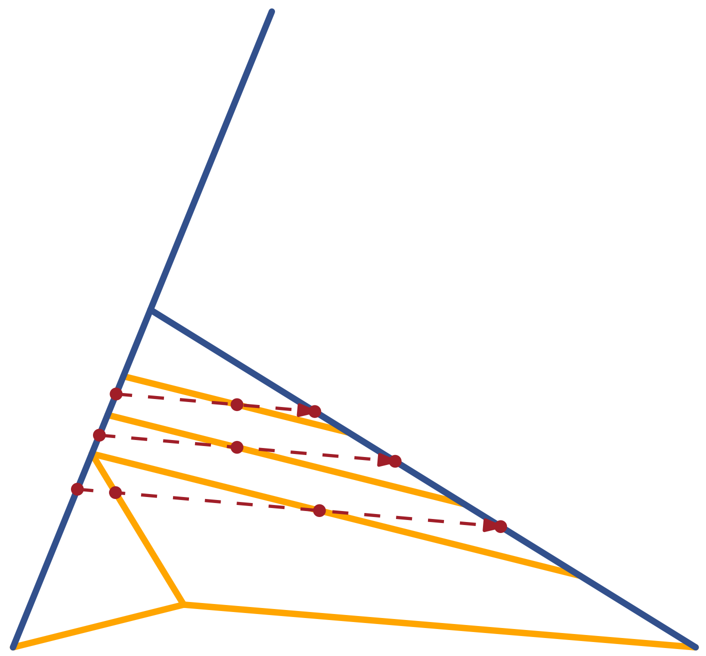

Repeatedly

- choose a long path and
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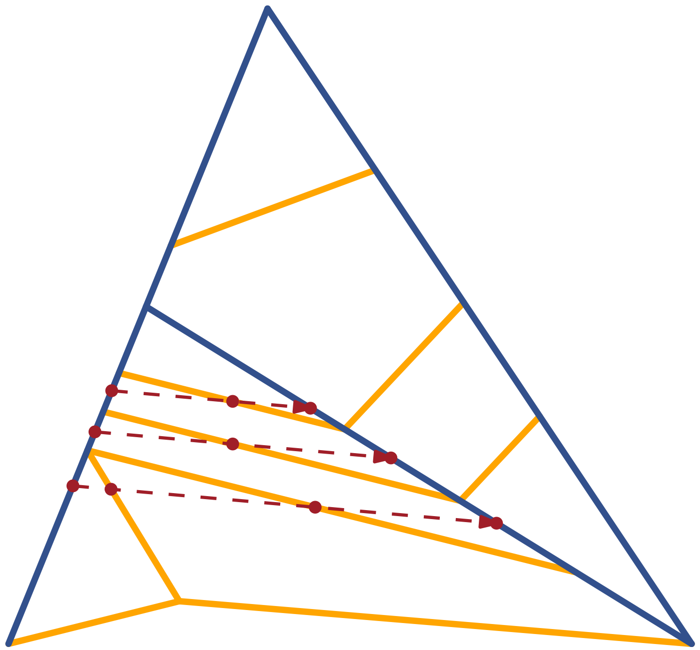

Repeatedly

- choose a long path and
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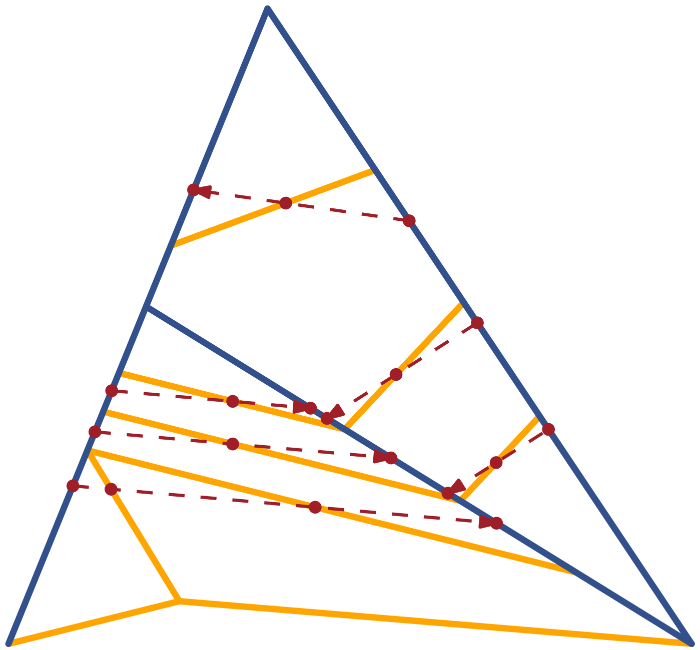
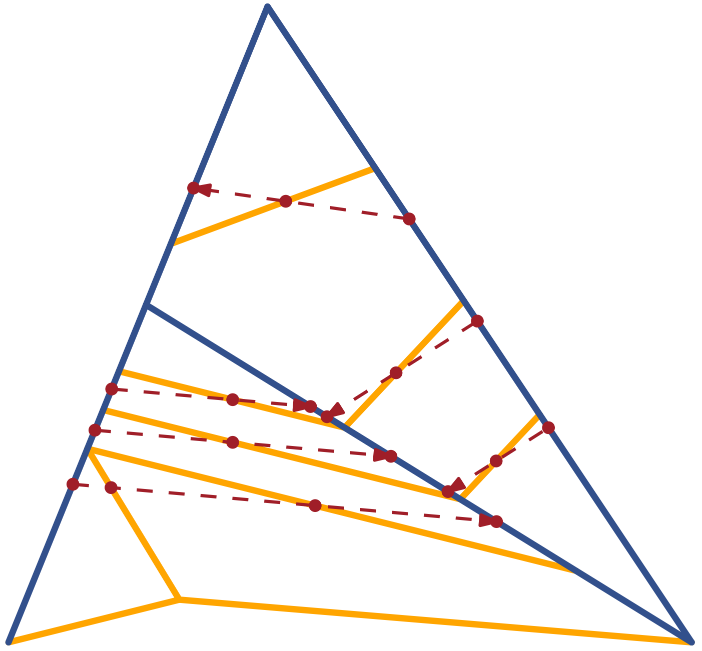


For any topological ordering
$O(n^{2/3} \log^{2/3}(n))$ pages suffice

Repeatedly

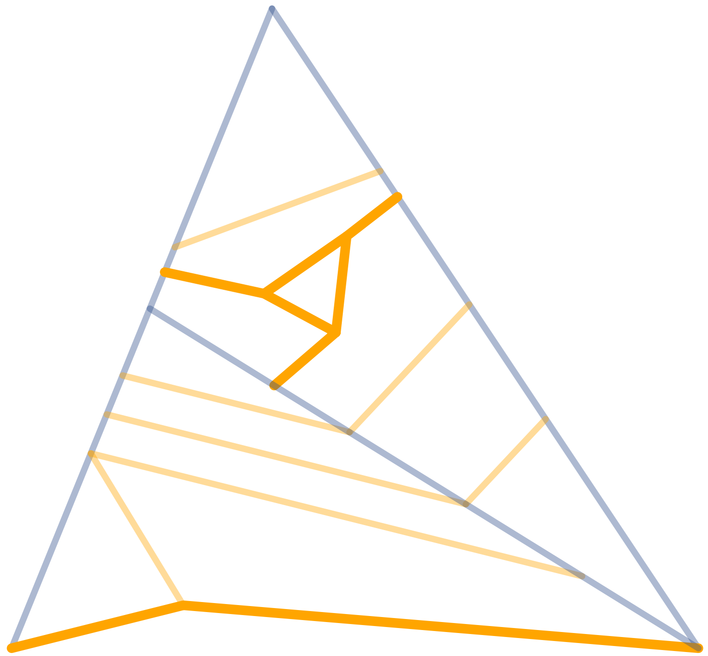
- choose a long path and
- apply width-theorem to subgraph of small width



For any topological ordering $O(n^{2/3} \log^{2/3}(n))$ pages suffice

Remaining subgraph

- no long paths
- apply height-theorem to subgraph of small height



Construct ordering s.t. $O(n^{2/3} \log^{2/3}(n))$ pages suffice for remaining edges

Page number of upward planar graphs

- lower bound: 5
- upper bound: $O(n^{2/3} \log^{2/3}(n))$

Page number of upward planar graphs

- lower bound: 5
- upper bound: $O(n^{2/3} \log^{2/3}(n))$

What is the maximum page number of

- upward planar graphs,
- planar posets,
- upward / acyclic outerplanar graphs?