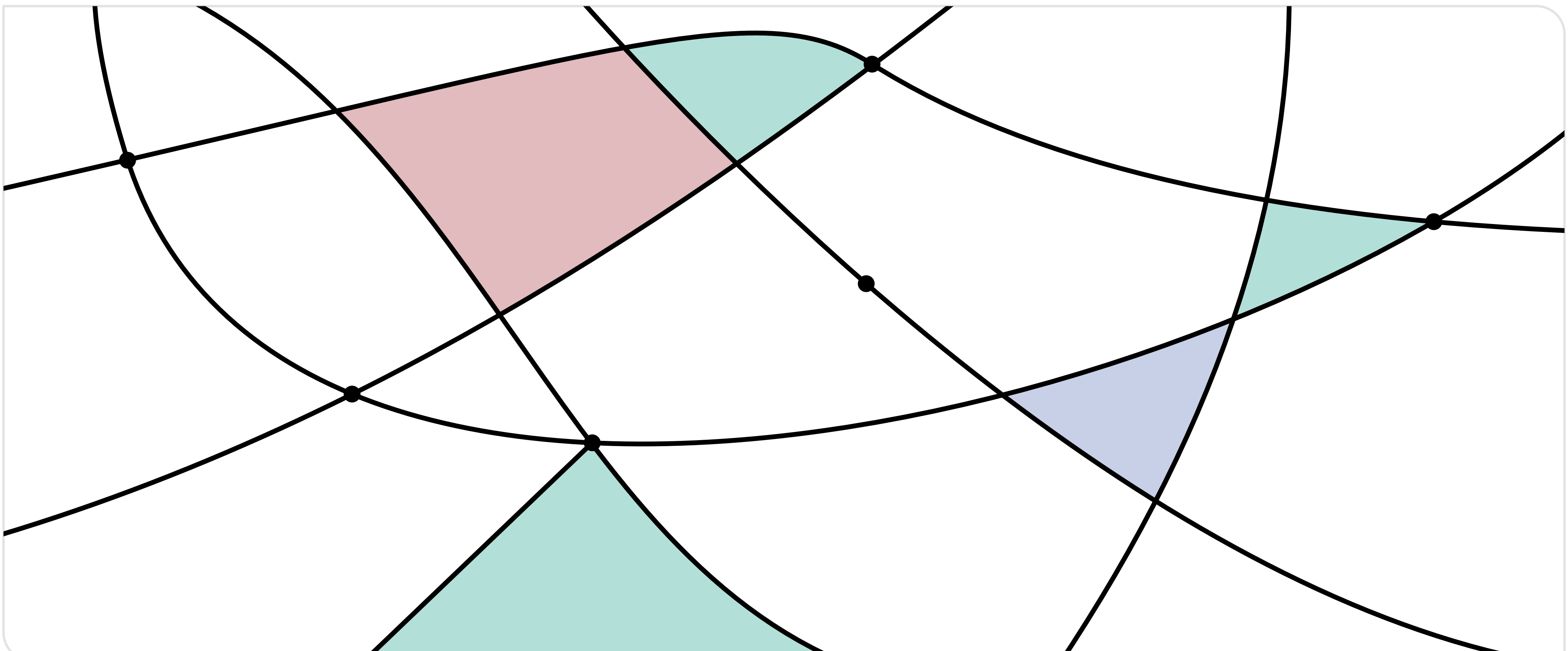
Karlsruher Institut für Technologie

Crossing Number of 3-Plane Drawings

joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt



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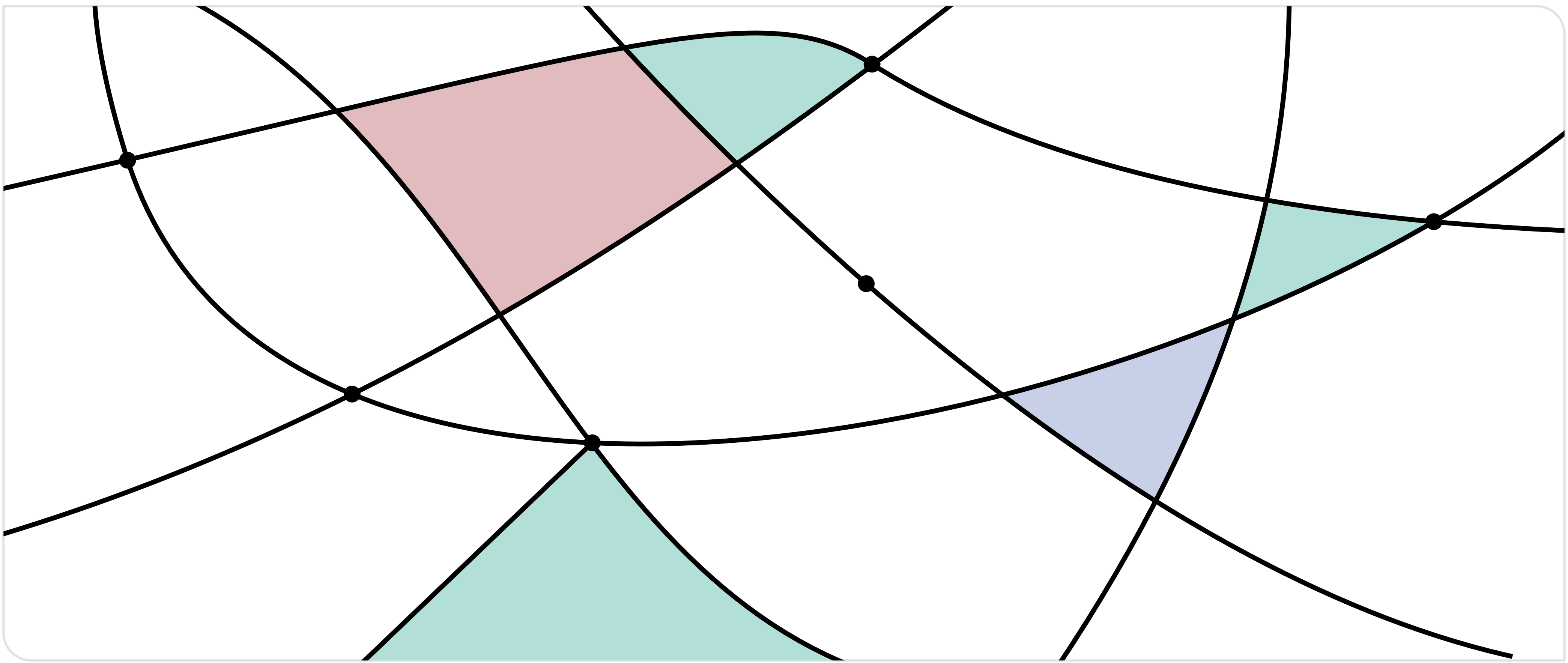
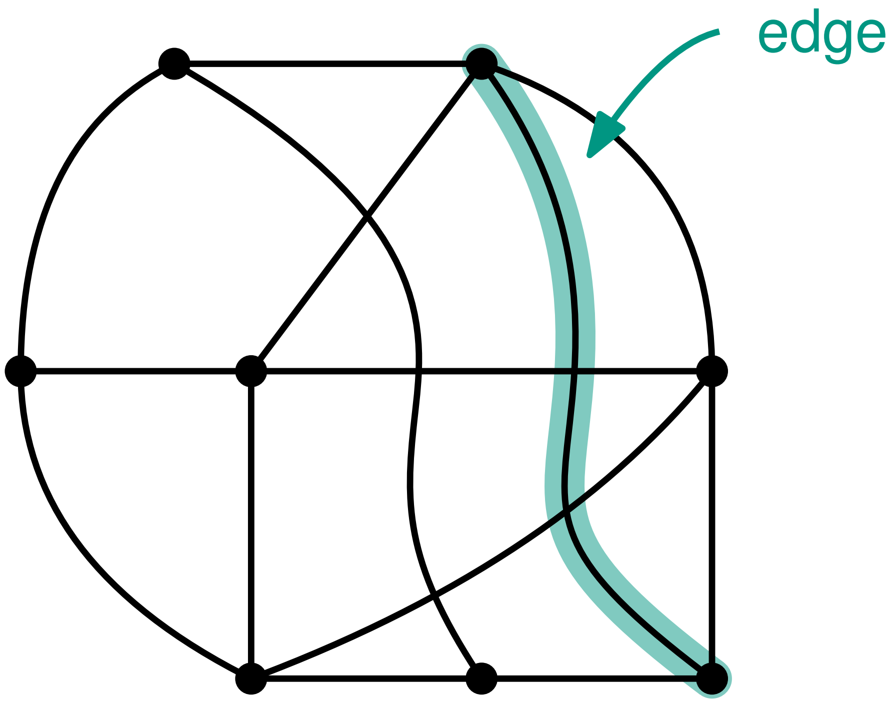
Crossing Number of ~~3-Plane~~ 2-Plane Drawings

joint work with Michael Hoffmann, Ignaz Rutter, Torsten Ueckerdt



Simple Drawings



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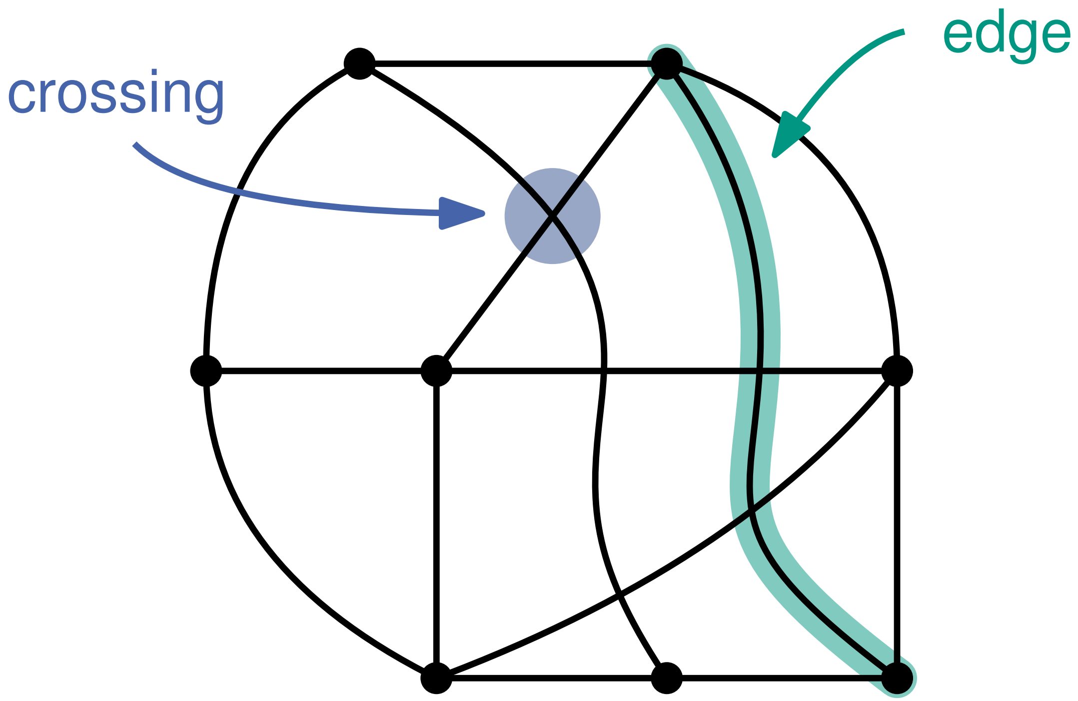

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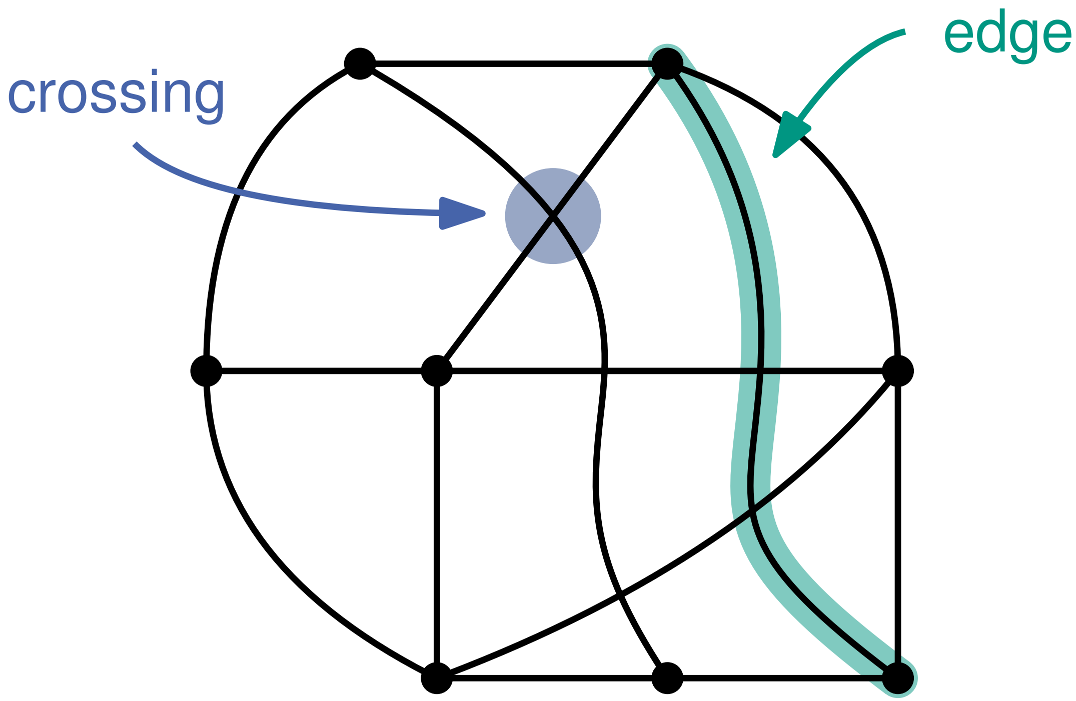

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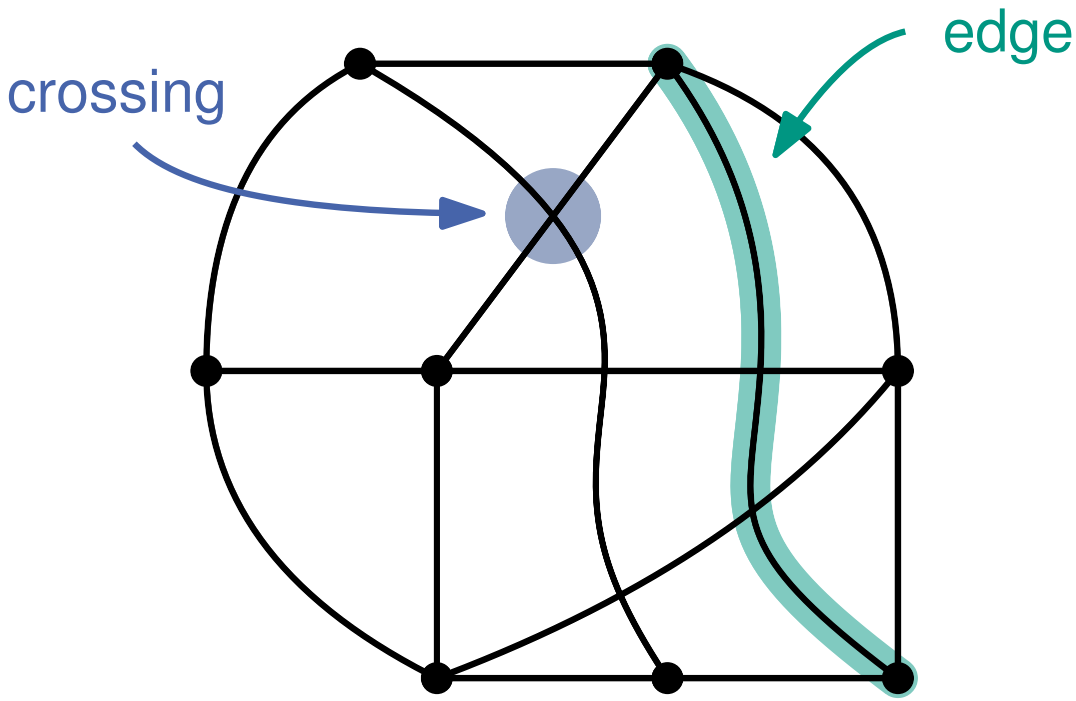

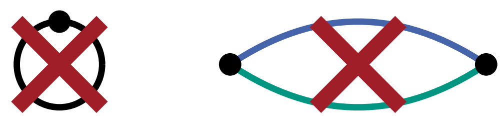

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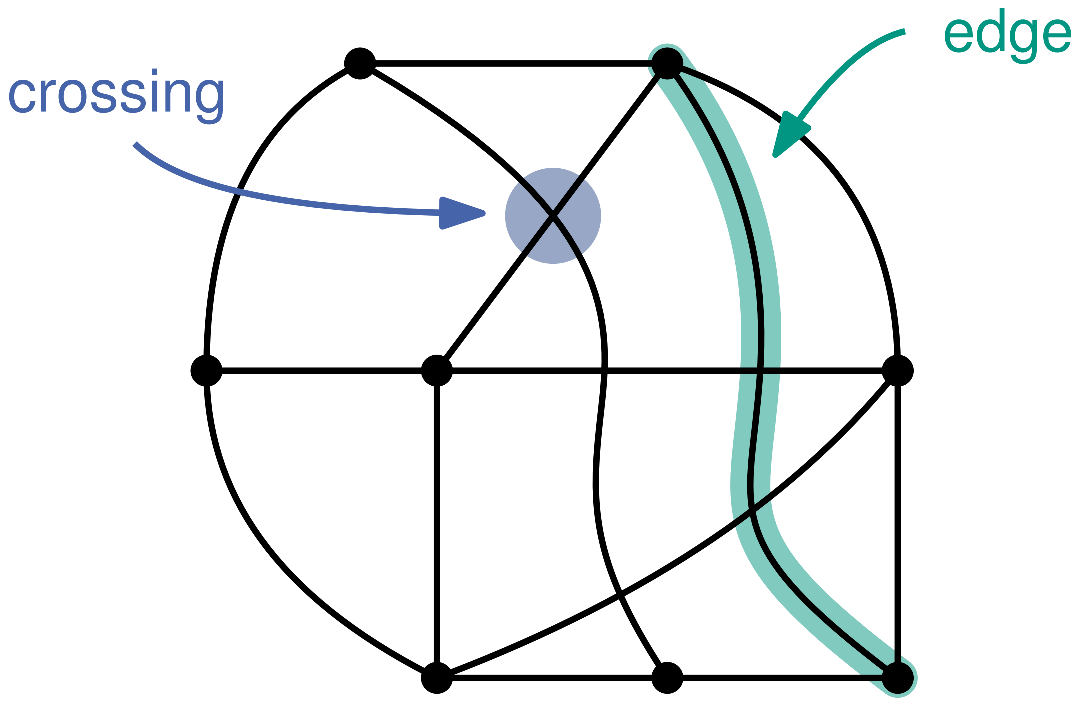

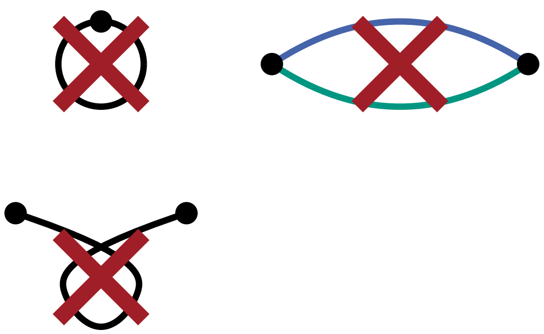

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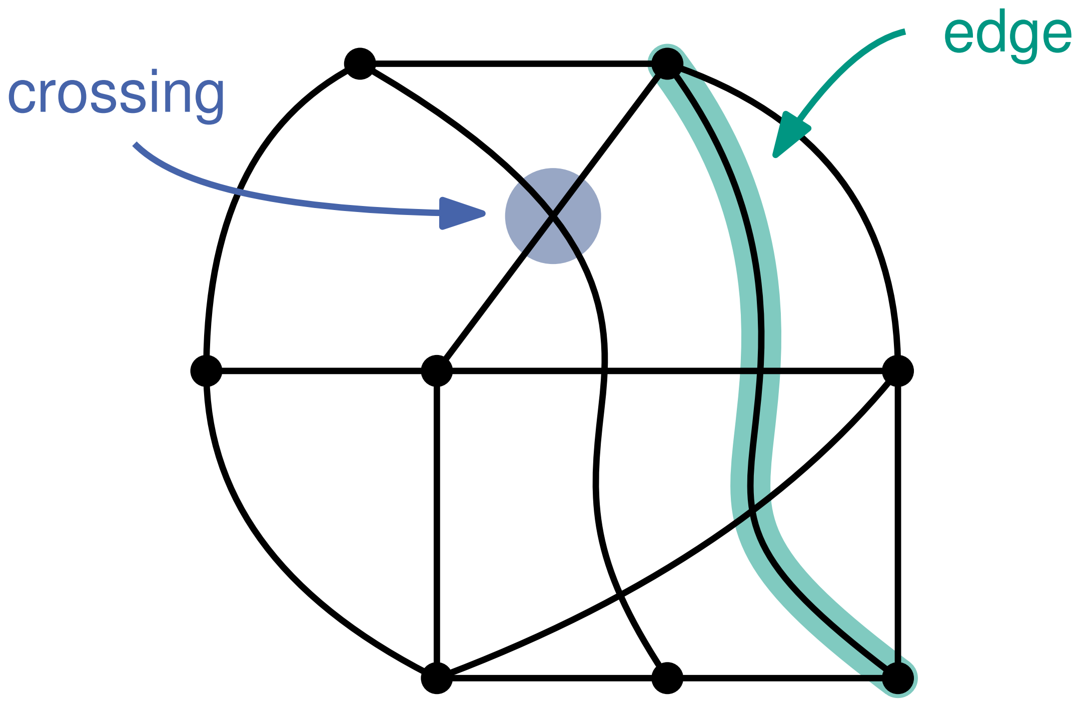

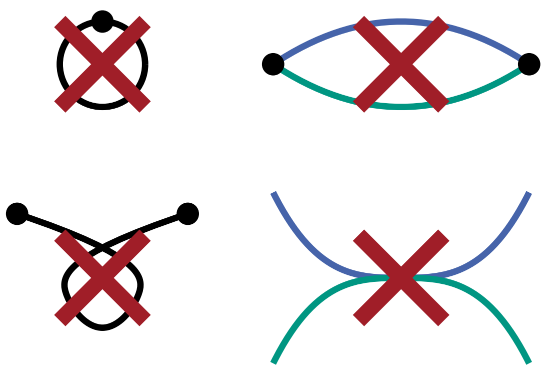

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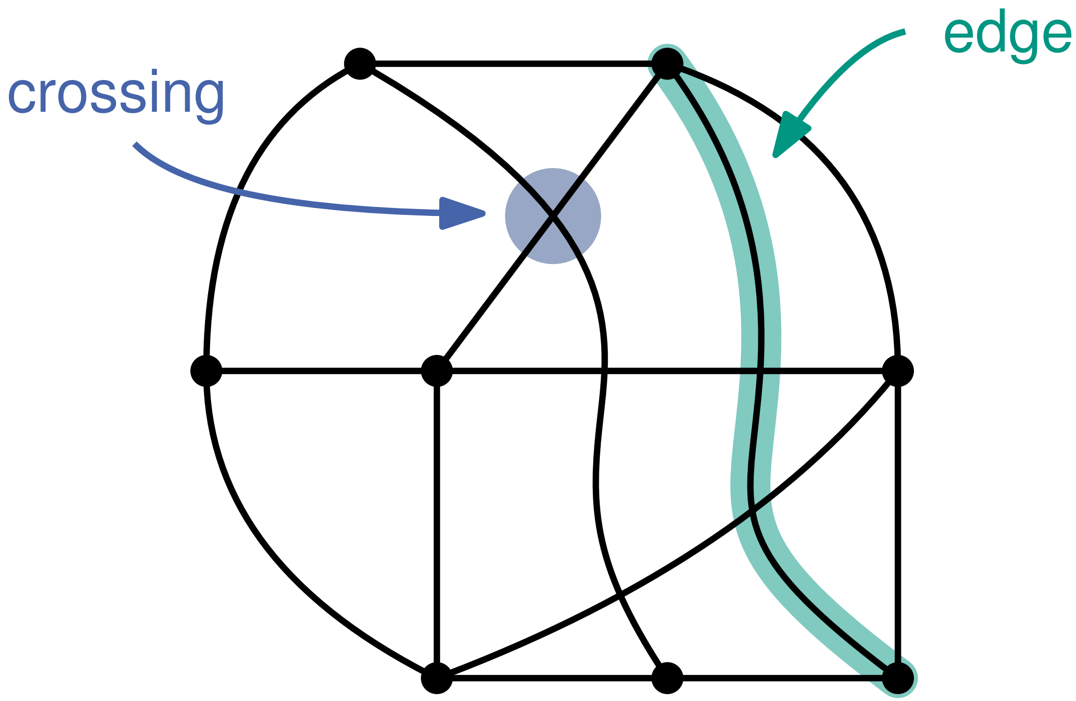

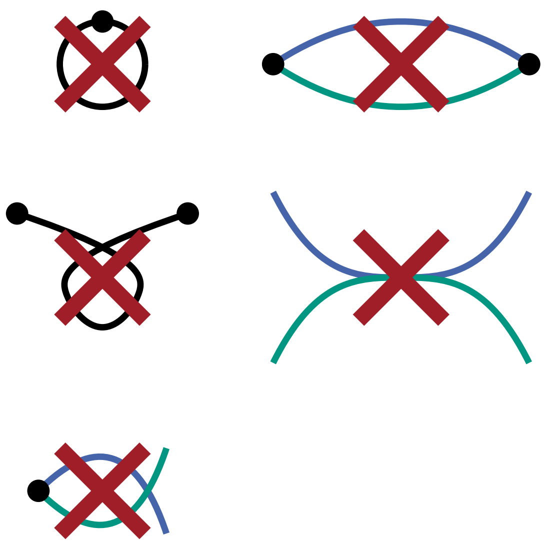

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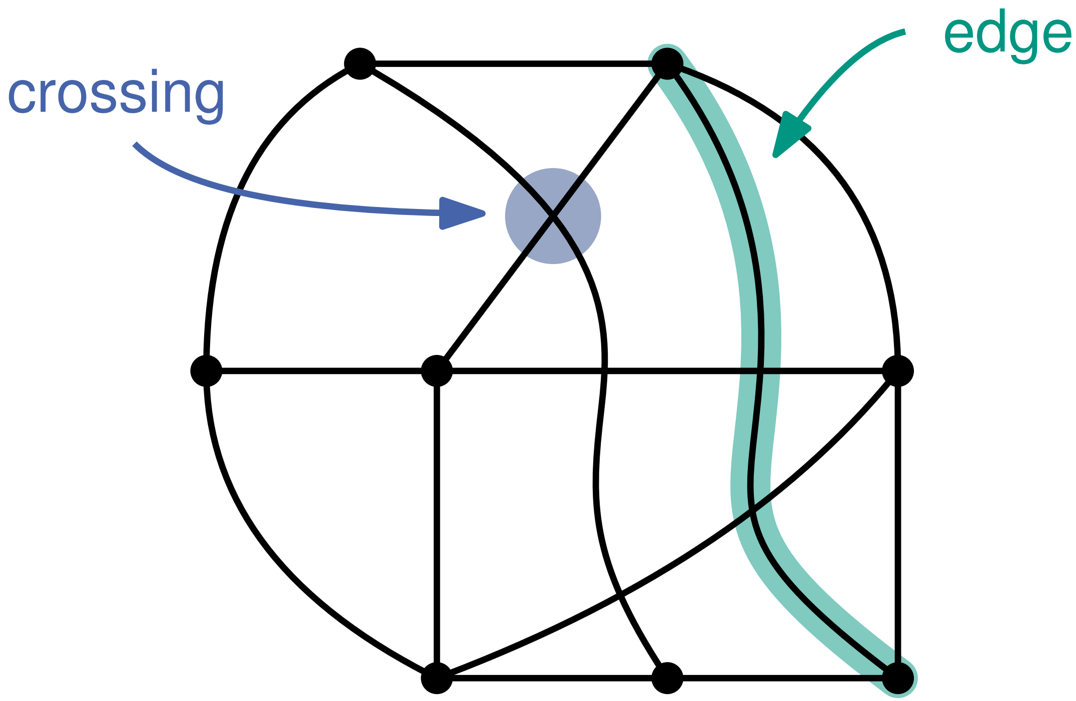

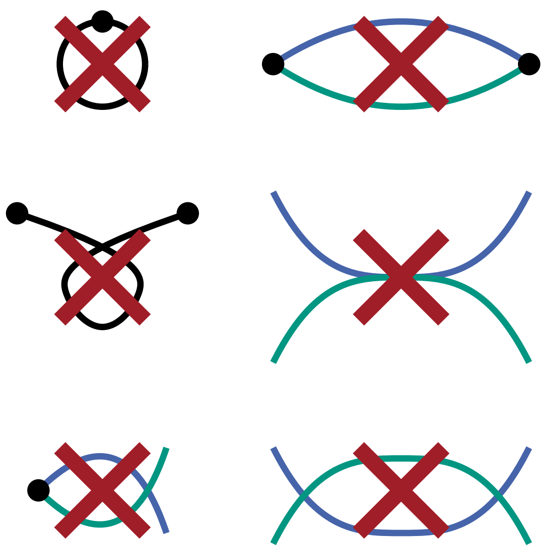

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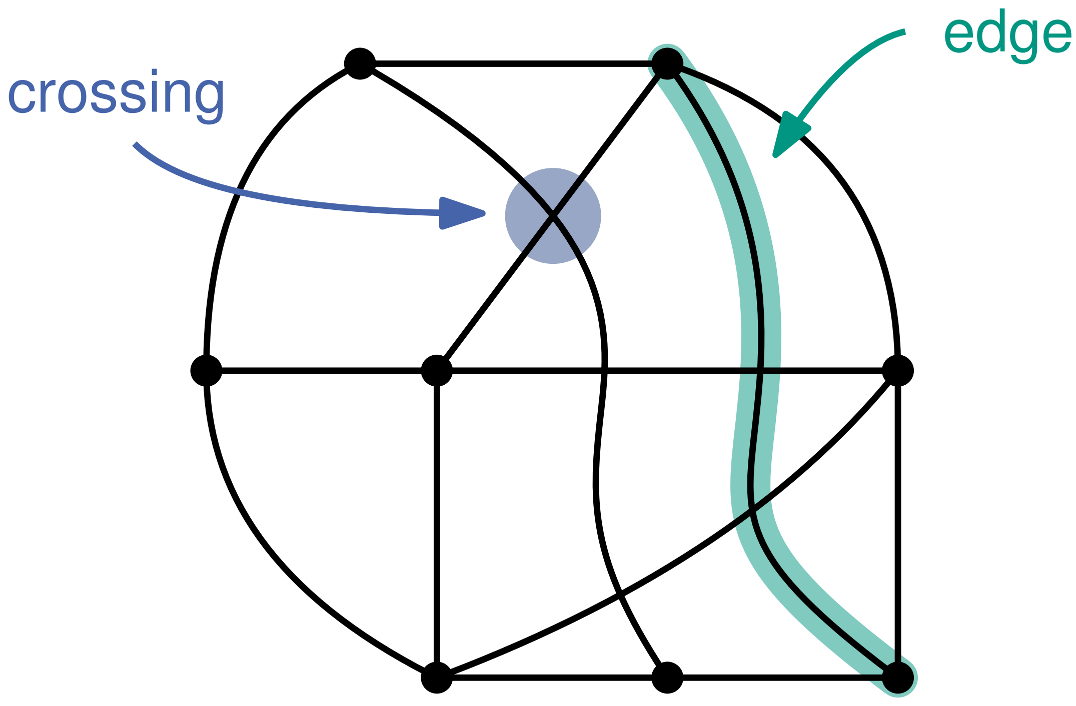

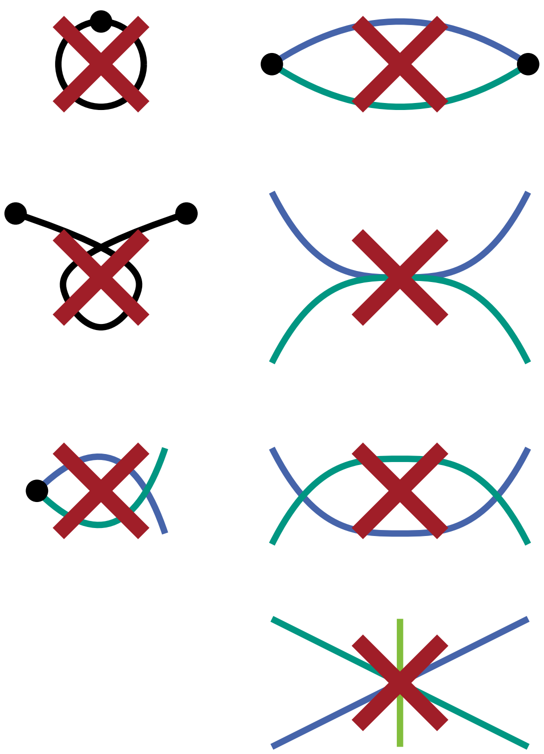

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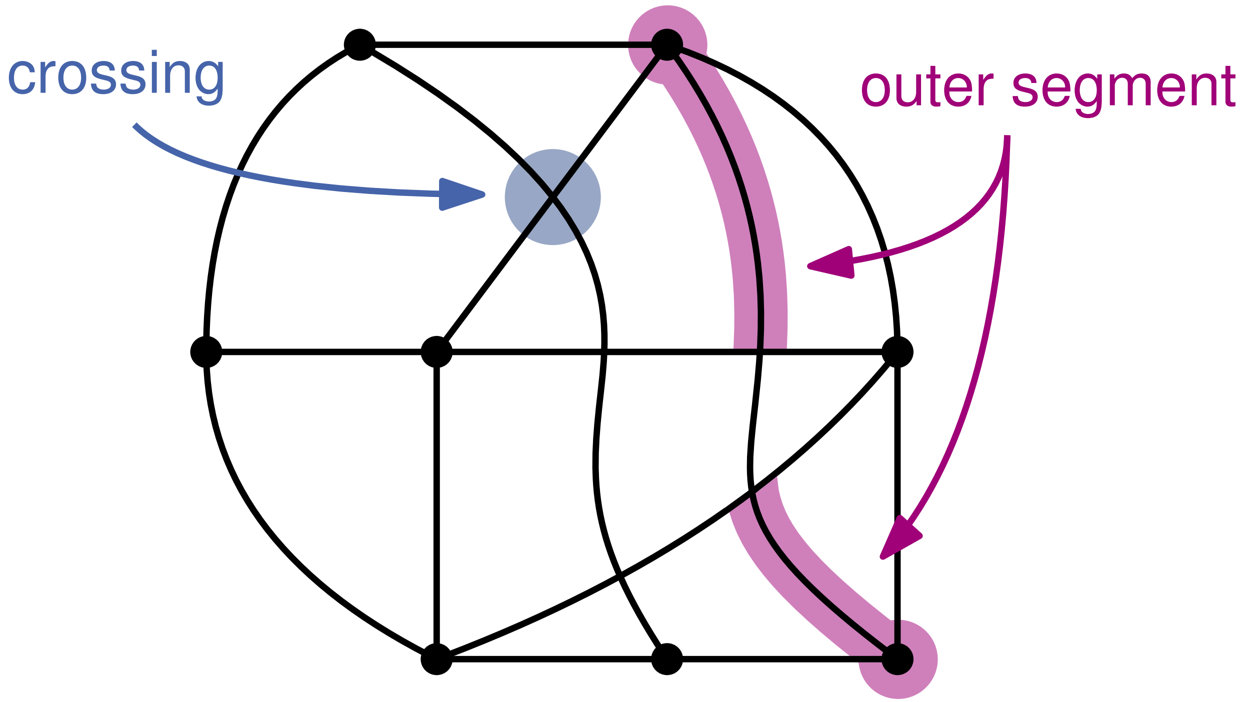

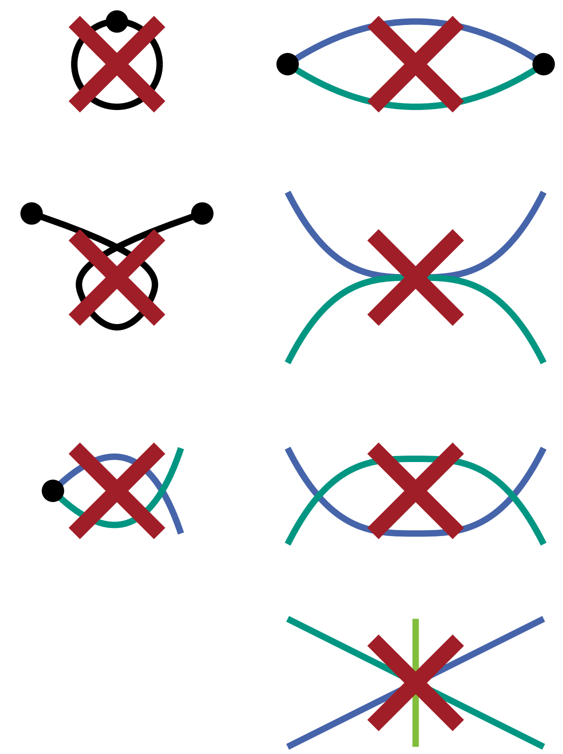

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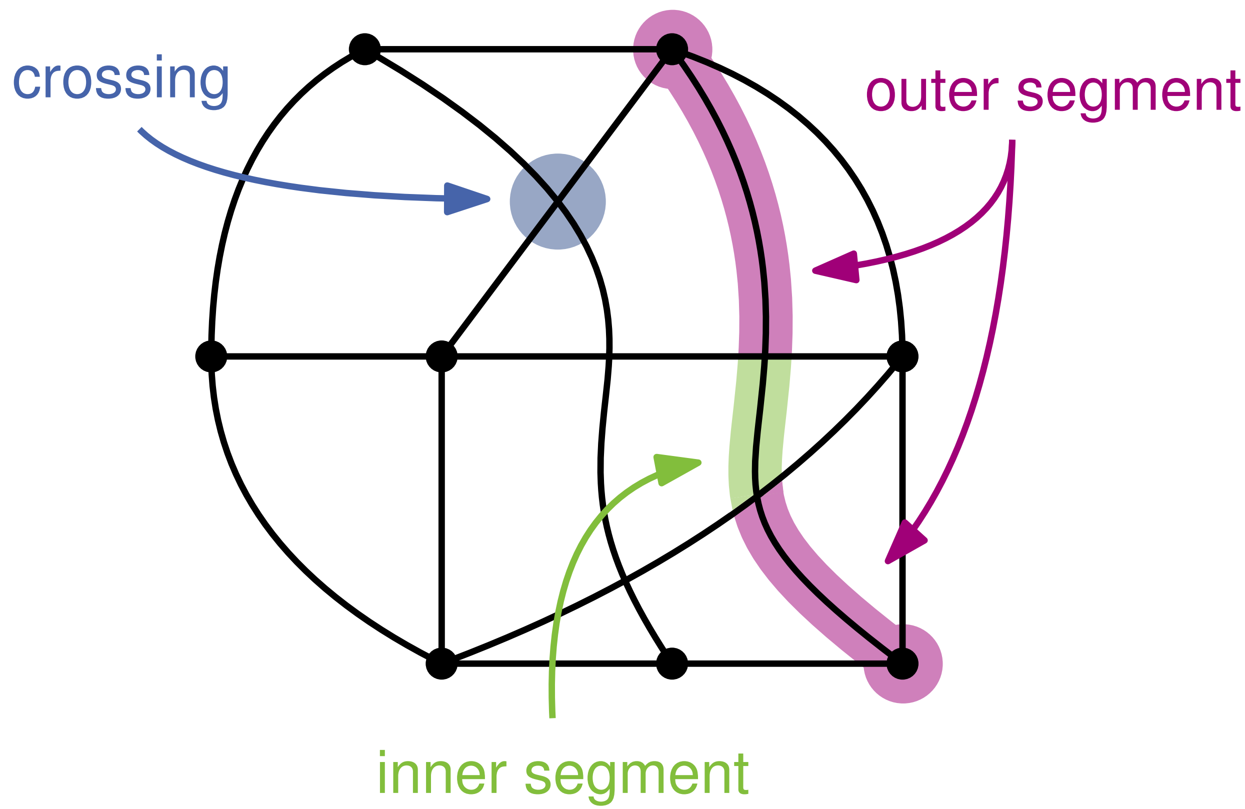

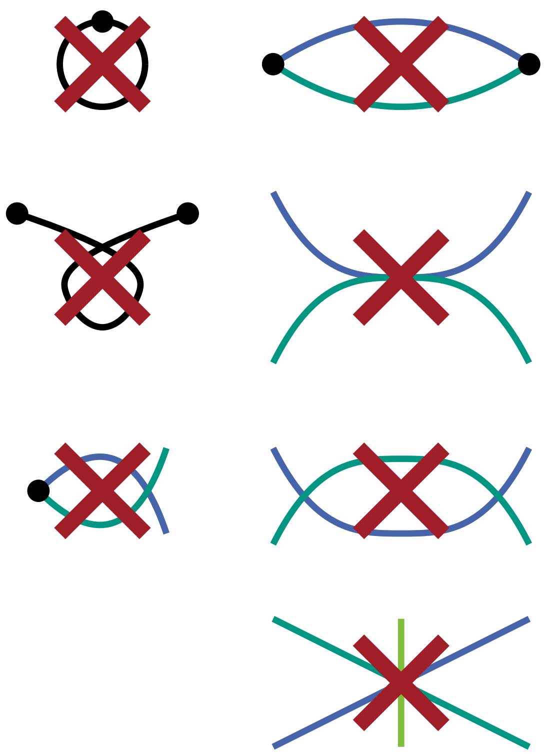

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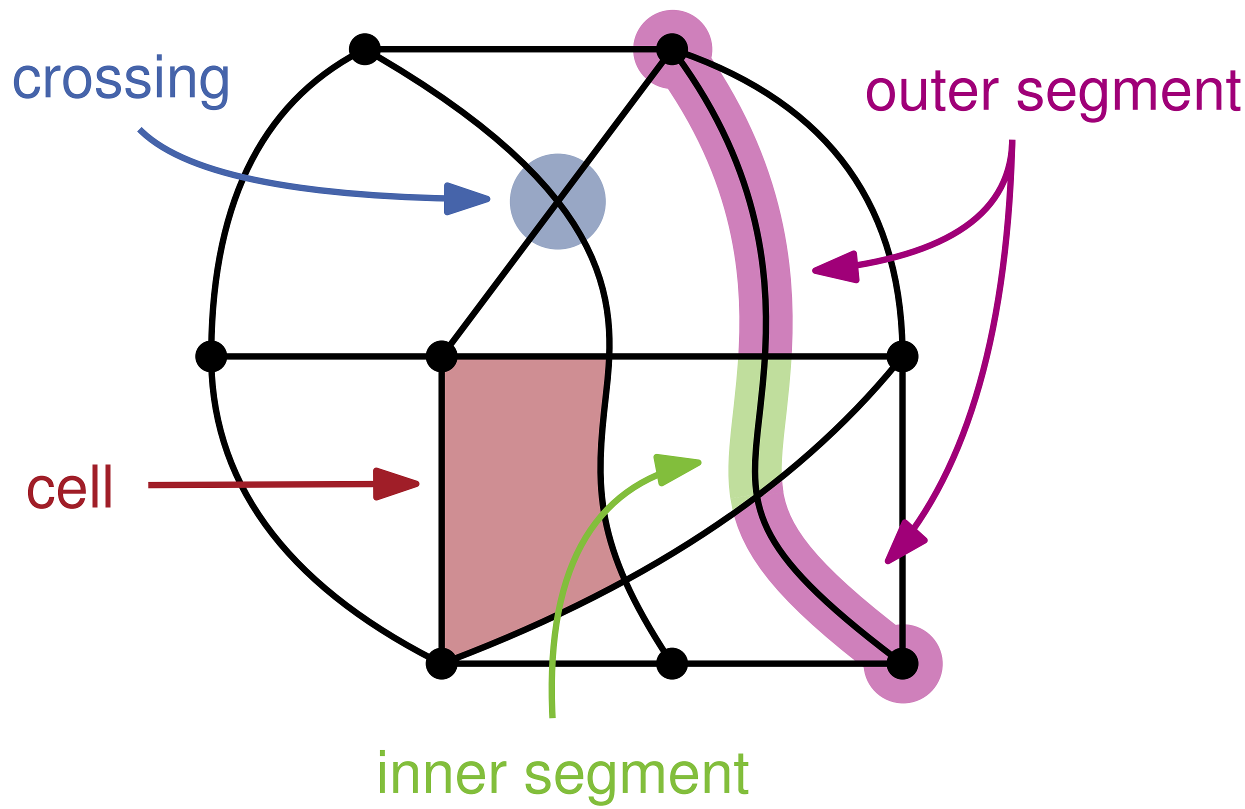
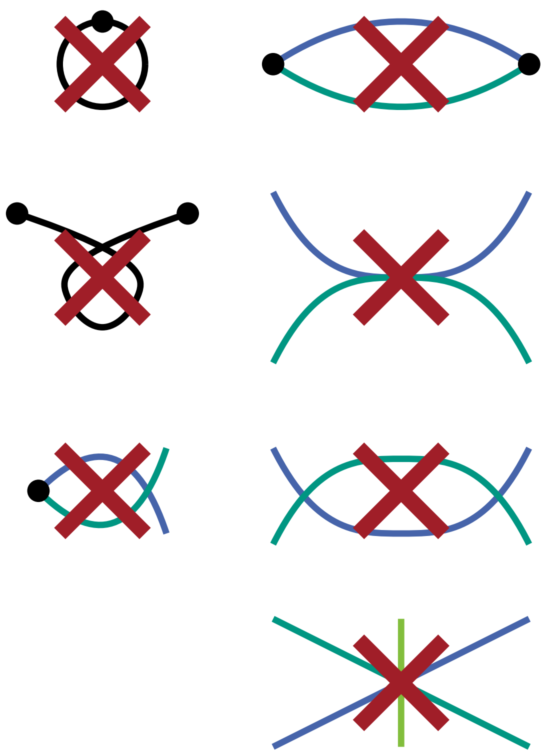
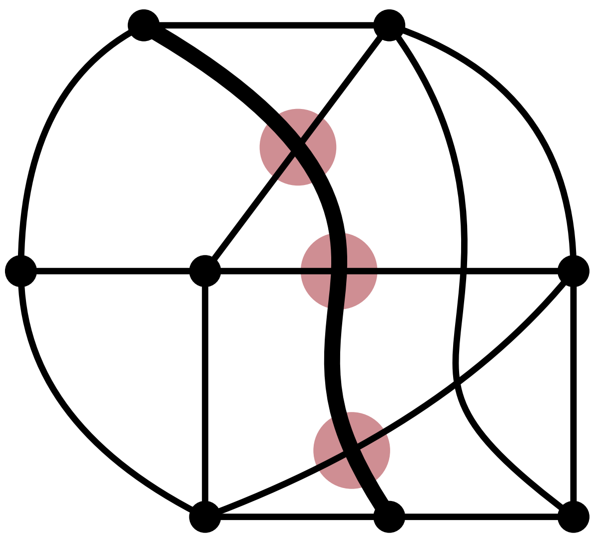




2-Plane Drawings

Def. A drawing is **2-plane** if every edge is crossed ≤ 2.

not 2-plane



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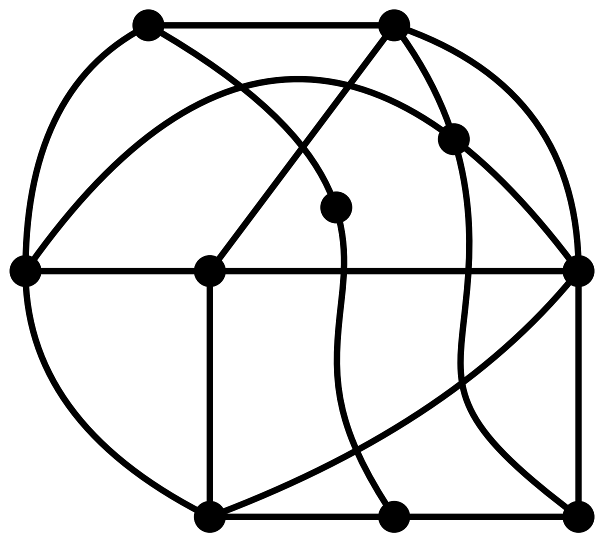

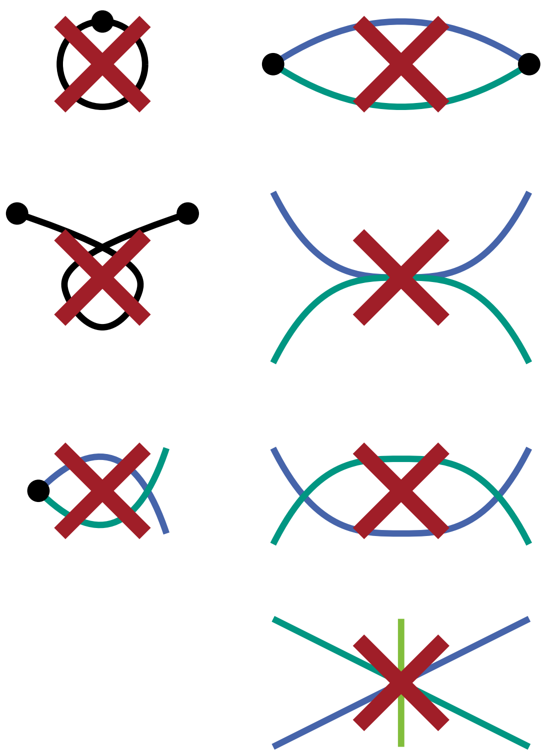

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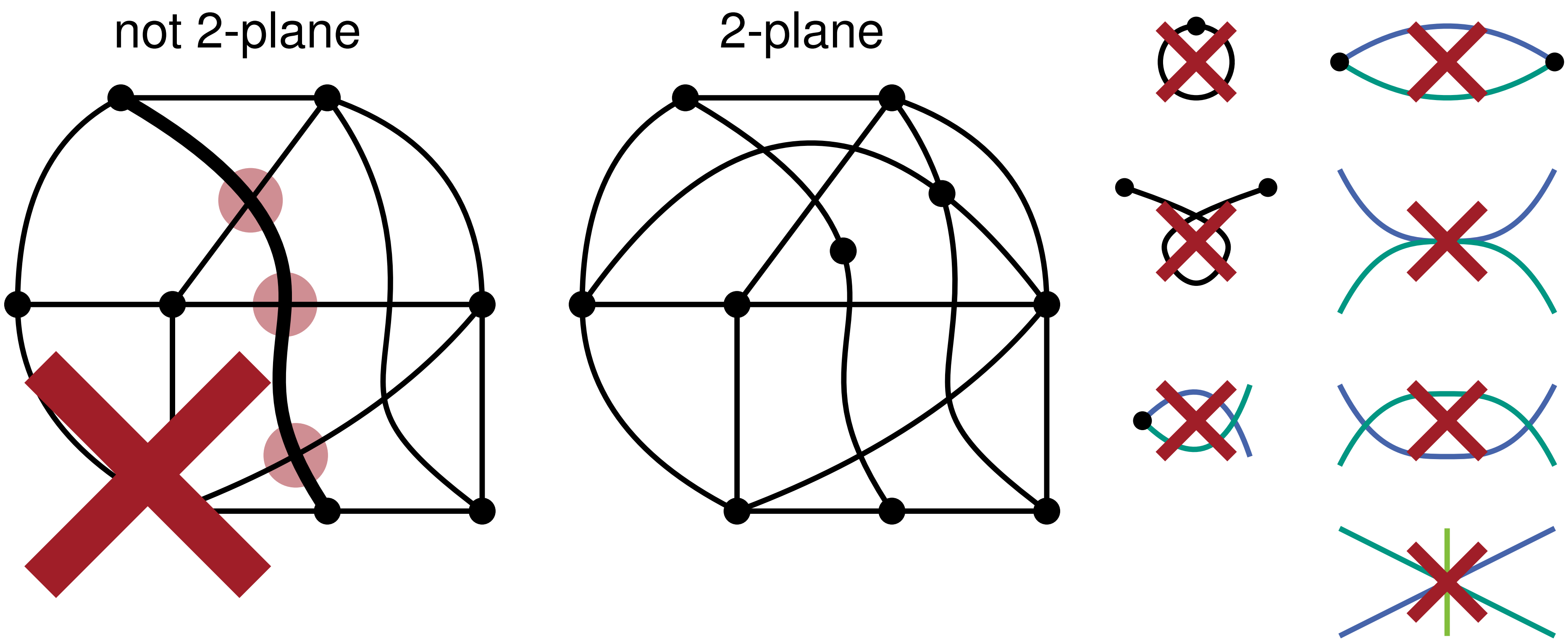

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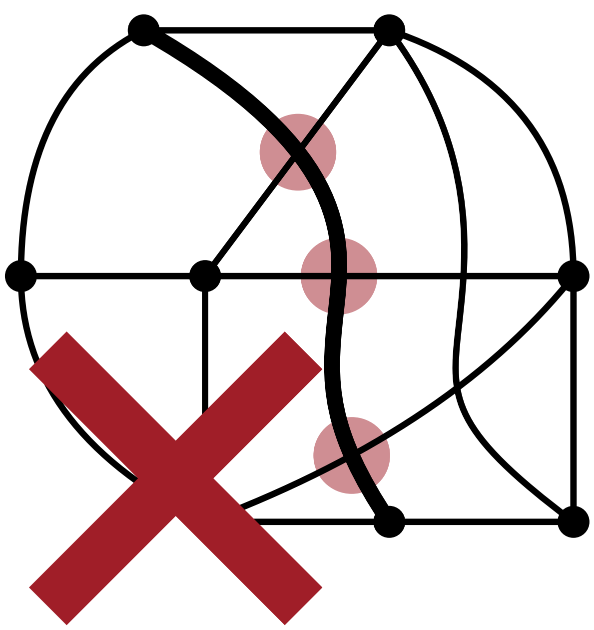

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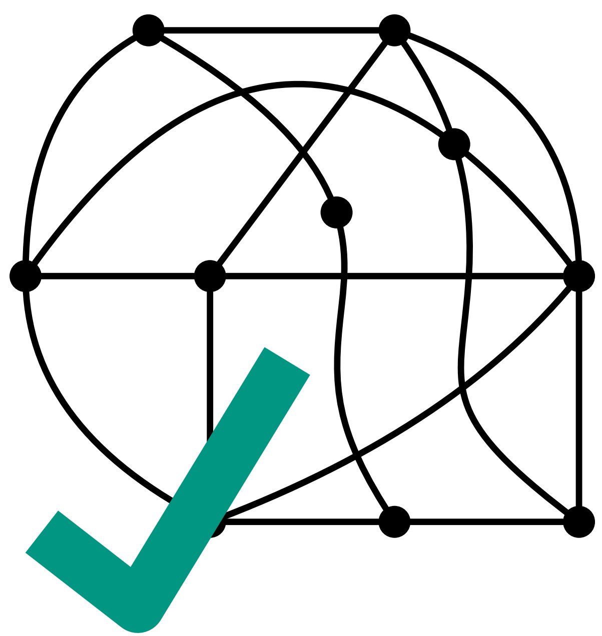
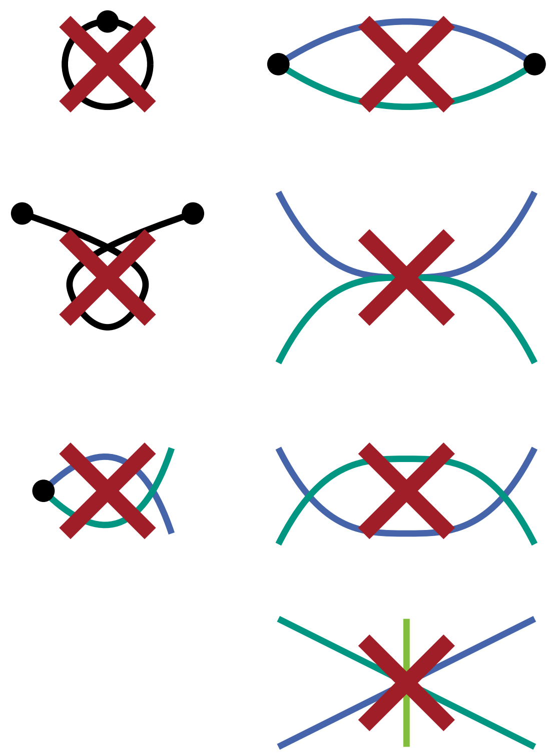


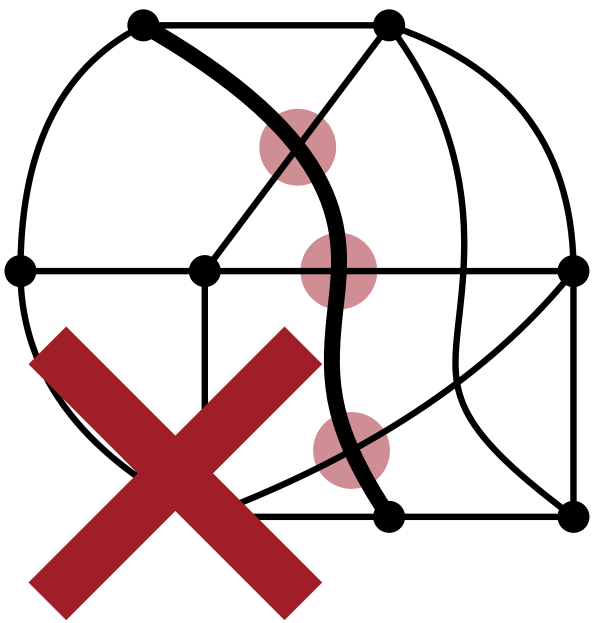


2-Plane Drawings

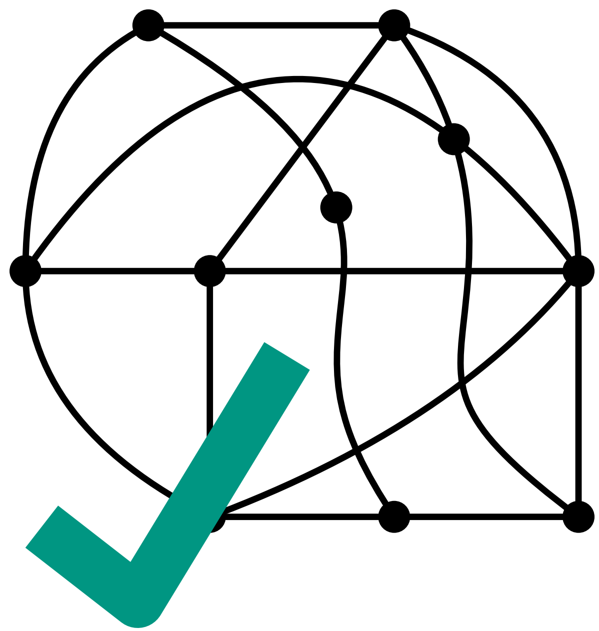
Def. A drawing is **2-plane** if every edge is crossed ≤ 2.

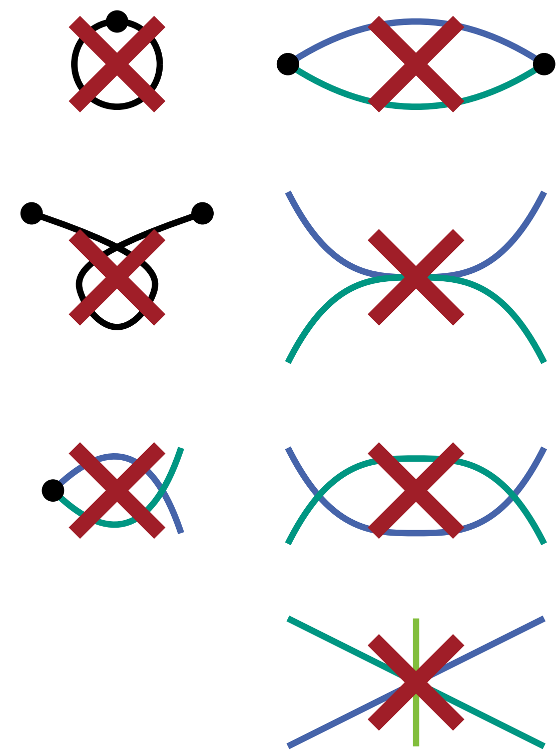
Question: How many crossings can a 2-plane drawing have?

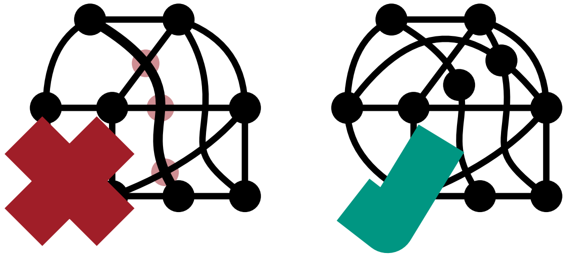
not 2-plane



2-plane







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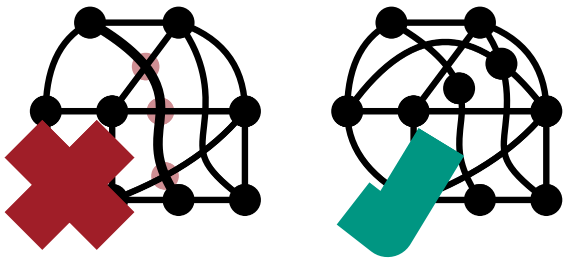

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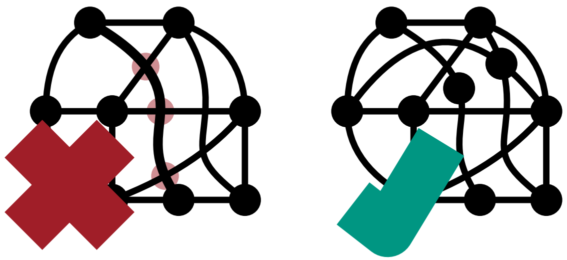
In terms of $m = |E|$:



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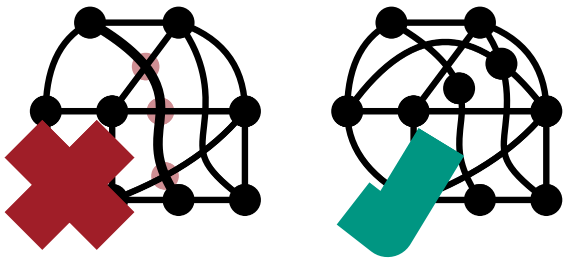
$$|\{(e,x) \mid x \in X, e \in E, x \text{ on } e\}|$$



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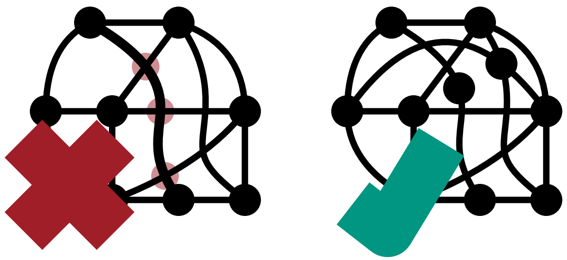

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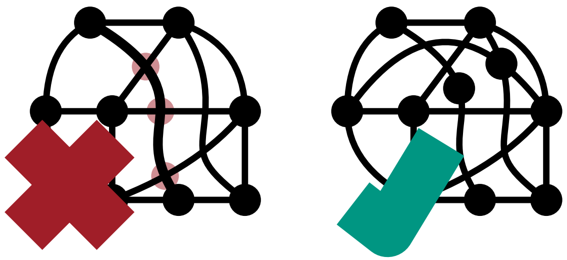
$$2|X| = |\{(e, x) \mid x \in X, e \in E, x \text{ on } e\}| \le 2m$$



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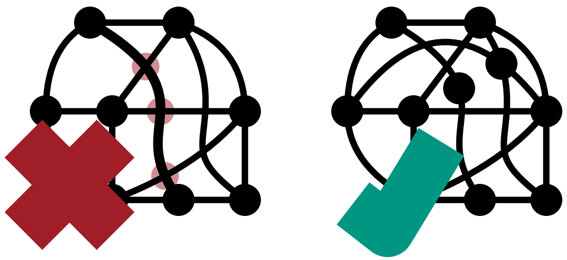

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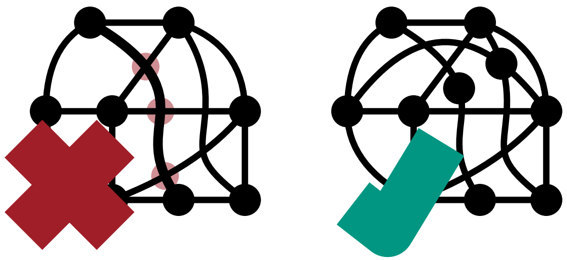

Question: How many crossings can a 2-plane drawing have?

Thm: $m \leq 5n - 10$.
(Pach, Tóth 1997)

In terms of $m = |E|$:

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$$|X| \leq m$$

In terms of $n = |V|$:

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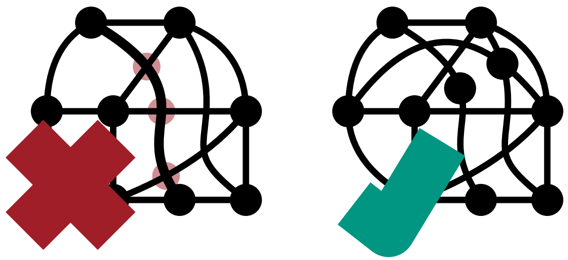

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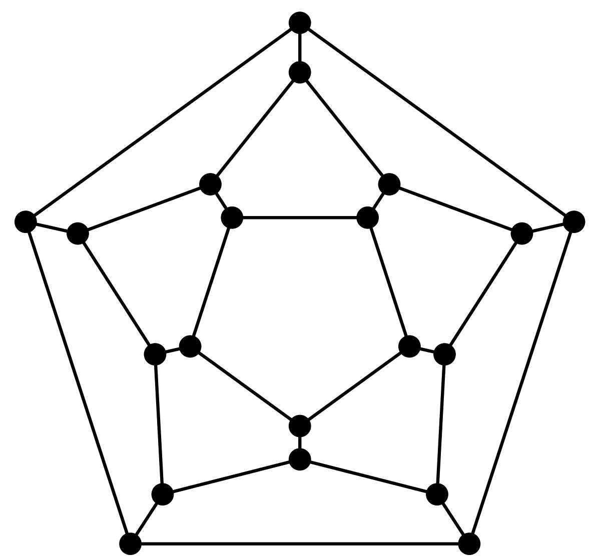

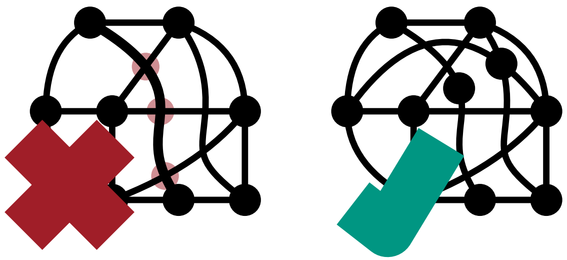

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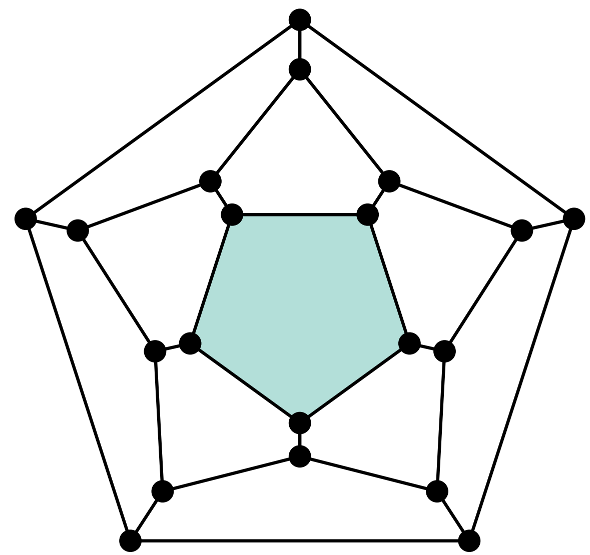

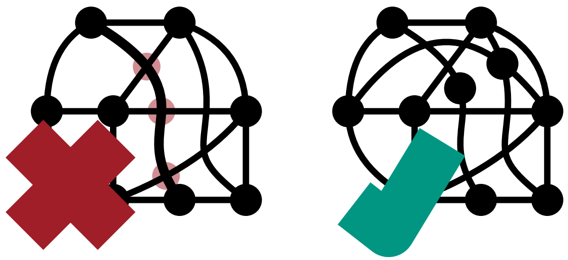

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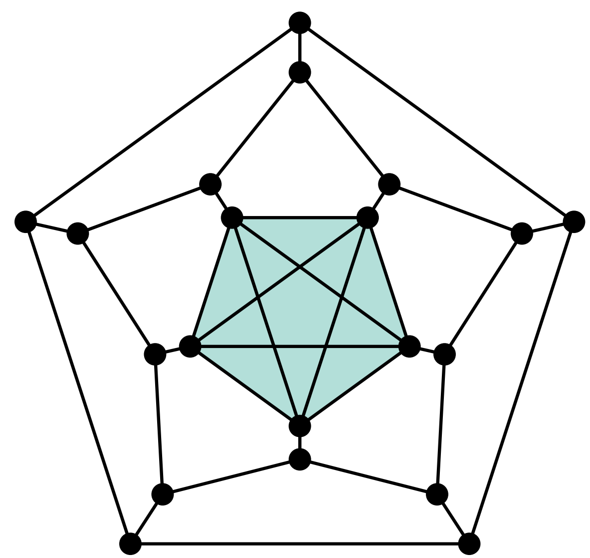

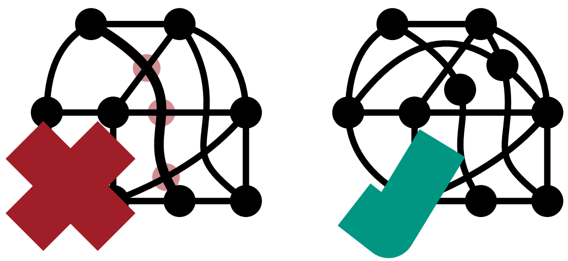

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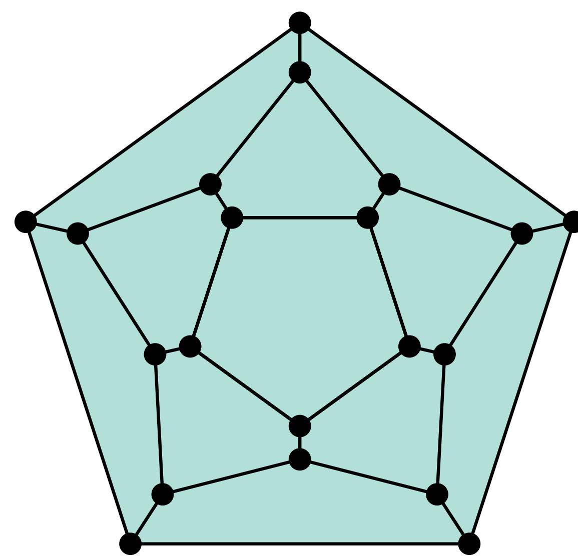

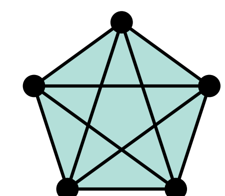

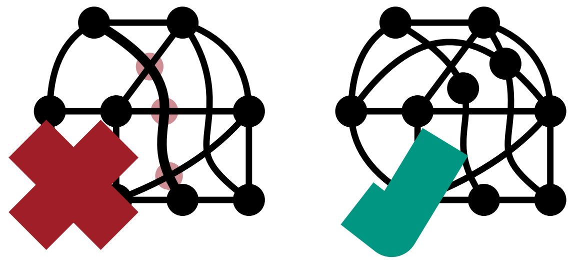

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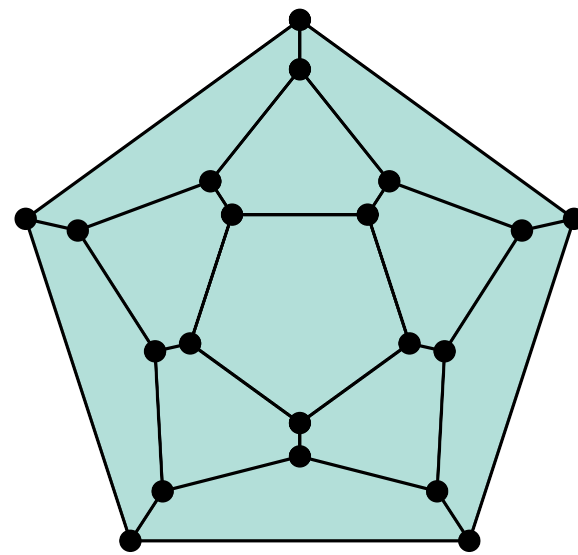
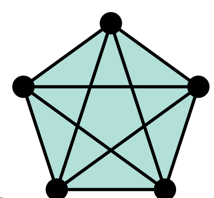
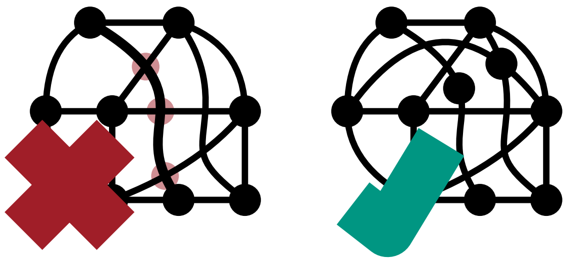




$$m = 5n - 10$$

$$|X| = \frac{10n - 4}{3}$$

$$\leq 3.\overline{3}n$$



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(Pach, Tóth 1997)

In terms of $n = |V|$:

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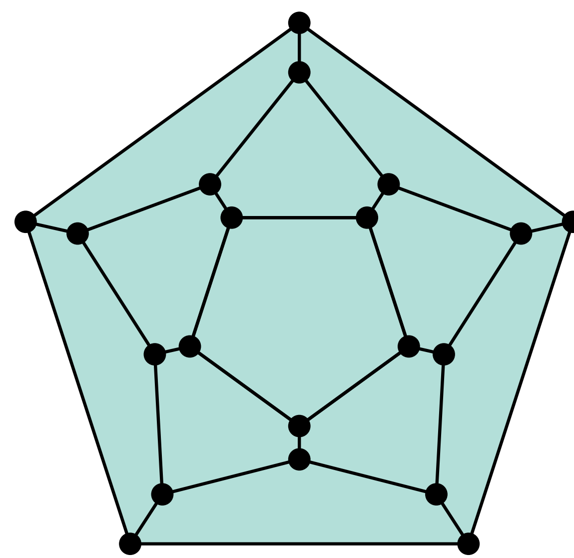
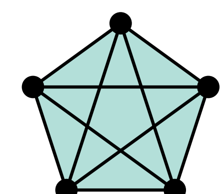




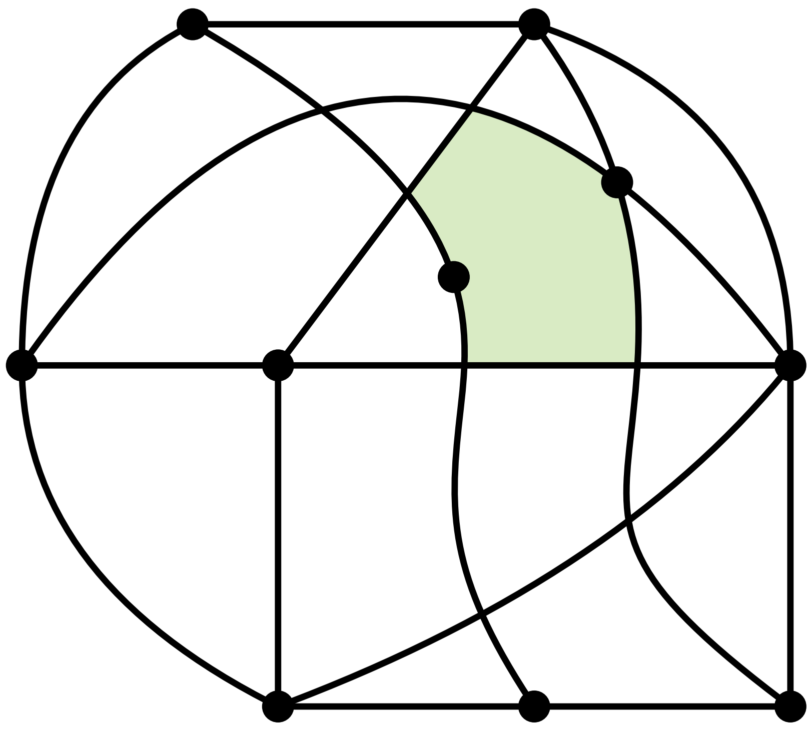
$$m = 5n - 10$$

$$|X| = \frac{10n-4}{3}$$

$$\leq 3.\overline{3}n$$

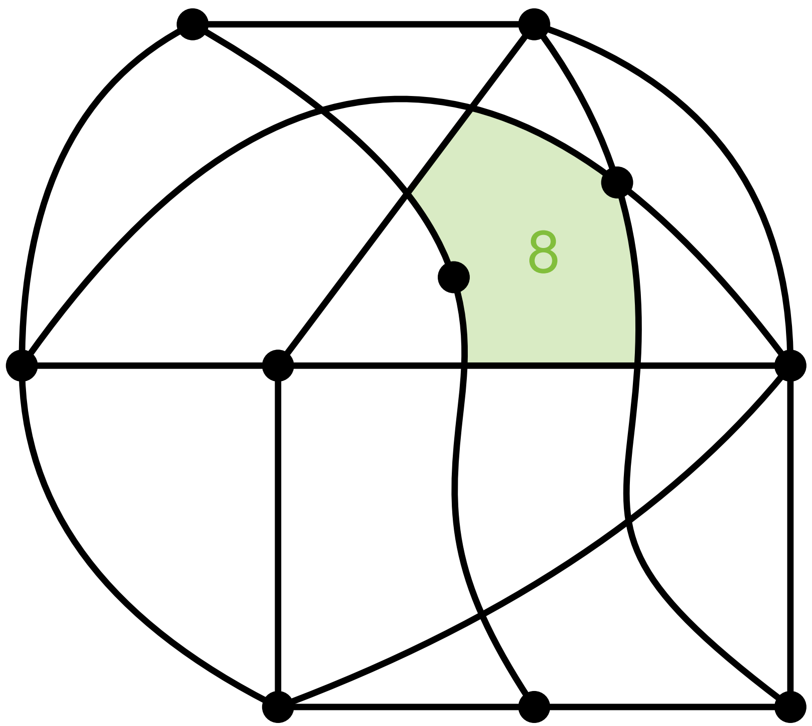
Question:
Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Sizes of Cells



$\|c\| = \#$ ●-inc $+$ #segment-inc

Sizes of Cells



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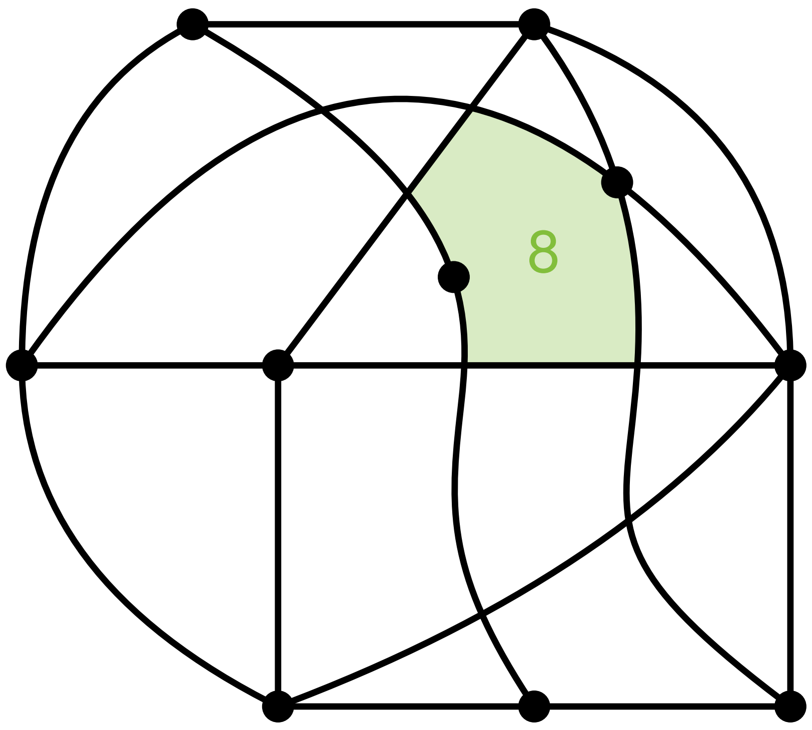

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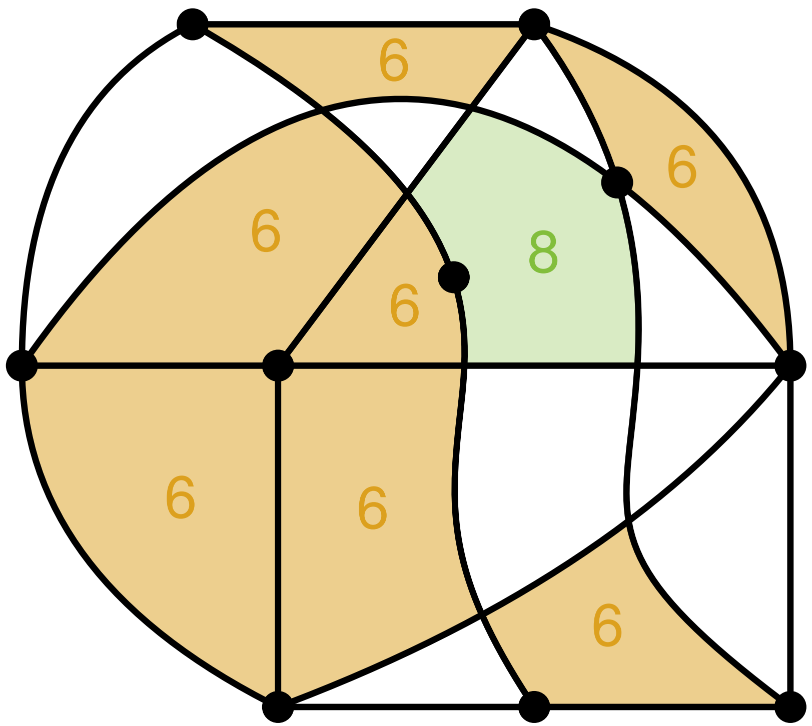
$\mathcal{C}_i = \{\text{cell } c\colon \|c\| = i\}$

Sizes of Cells

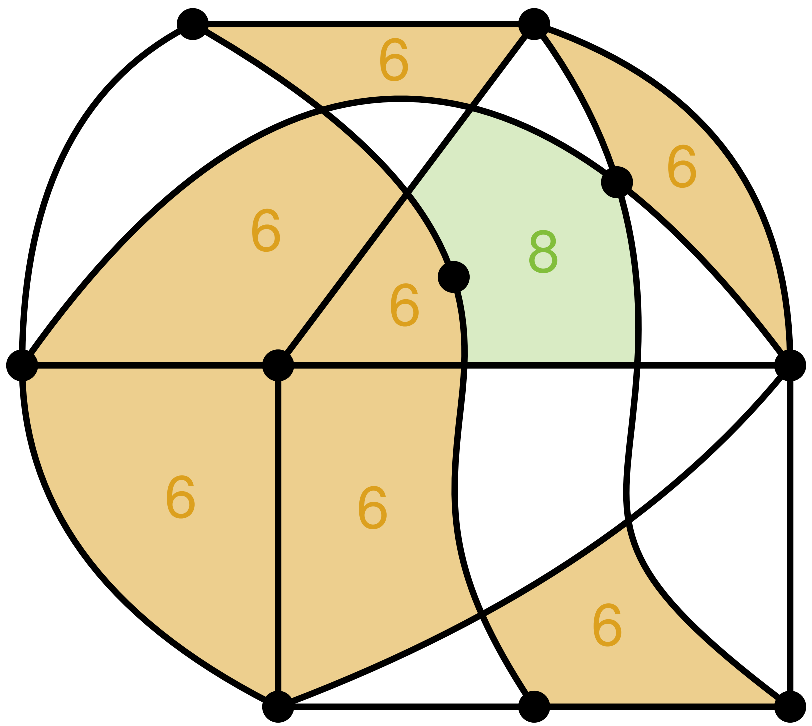


$\|c\| = \#\,\bullet\text{-inc} + \#\text{segment-inc}$

$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$

$\mathcal{C}_6$

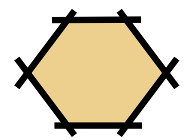
Sizes of Cells



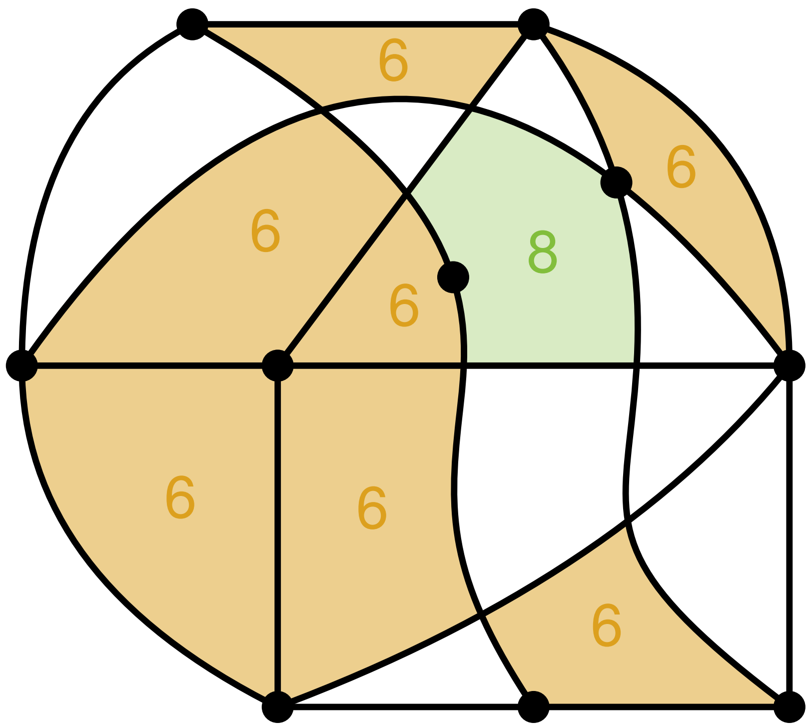
$\|c\| = \#\bullet\text{-inc} + \#\text{segment-inc}$

$\mathcal{C}_i = \{\text{cell } c \colon \|c\| = i\}$

$\mathcal{C}_6$



Sizes of Cells



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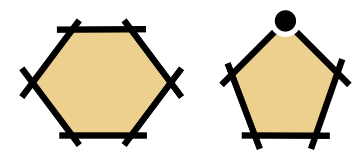
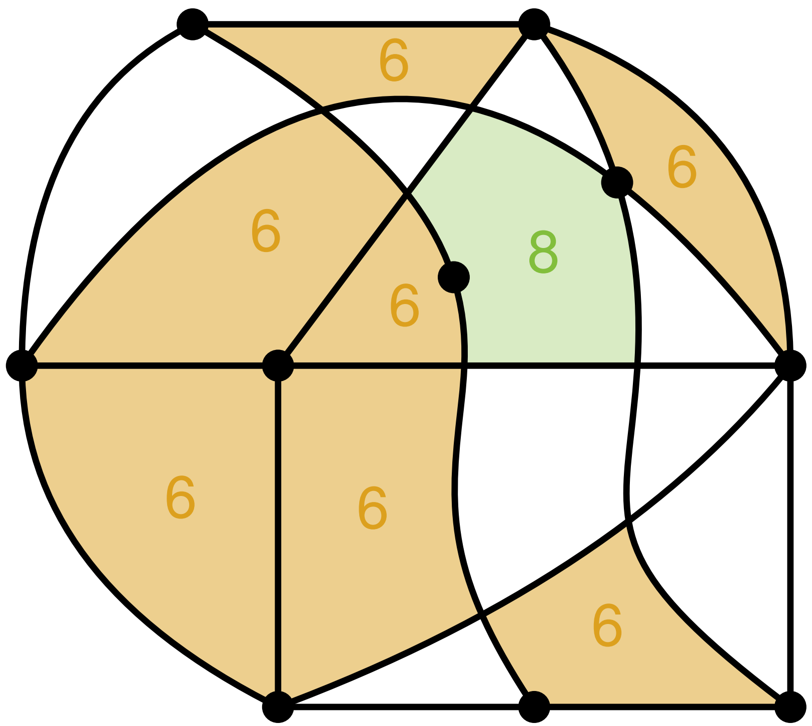


Sizes of Cells



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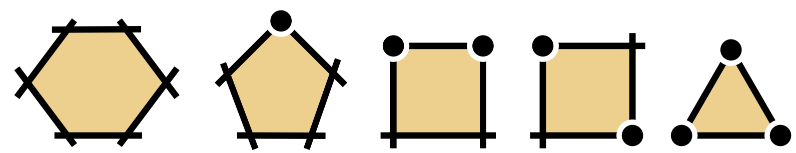
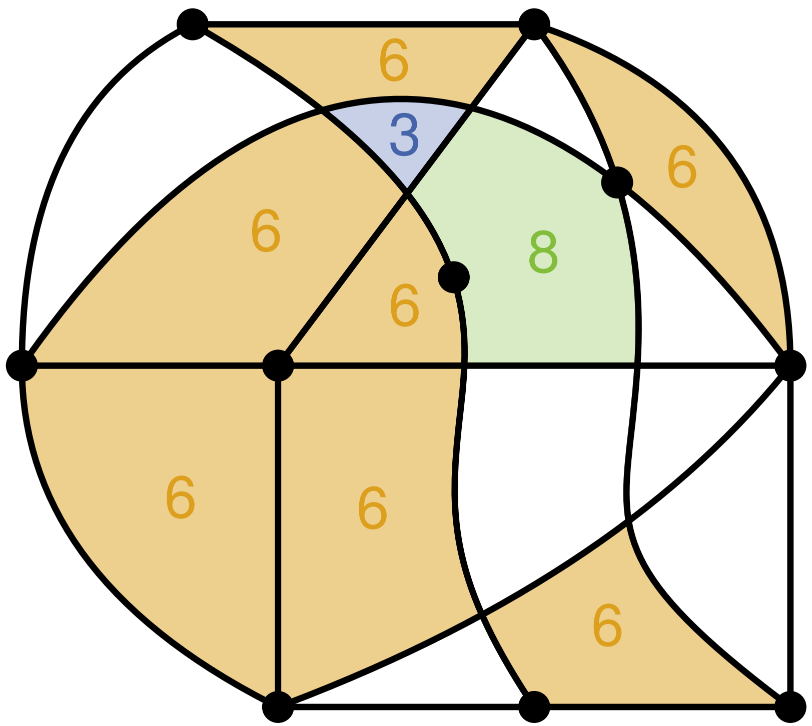
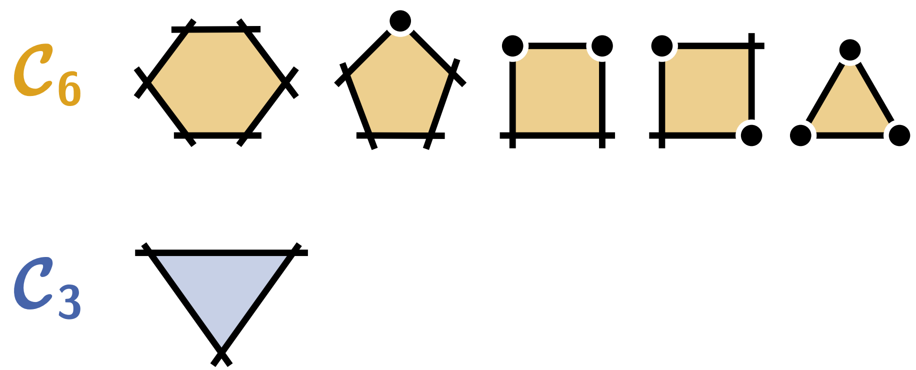


Sizes of Cells

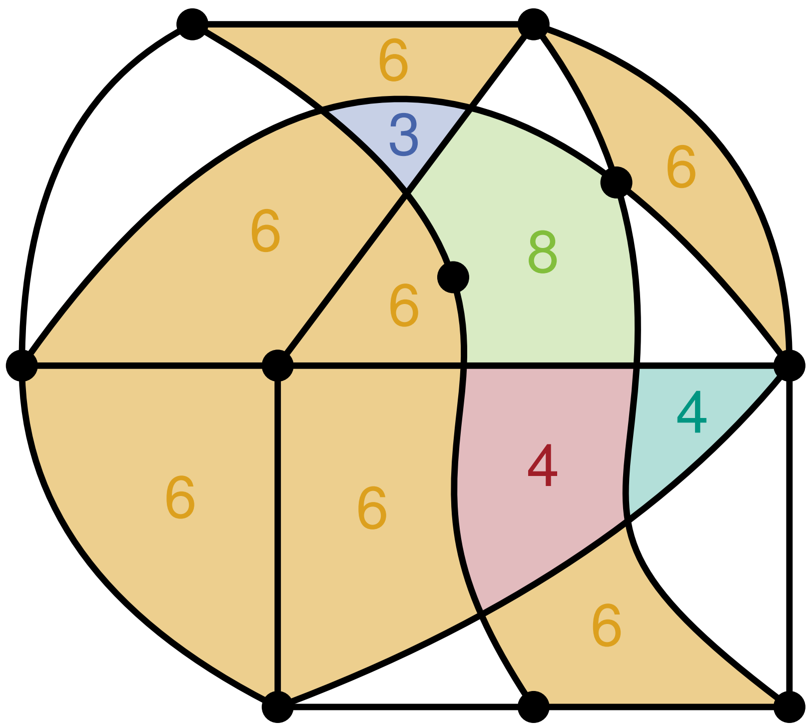


$\|c\| = \#\ \bullet\text{-inc} + \#\text{segment-inc}$

$\mathcal{C}_i = \{\text{cell } c : \|c\| = i\}$



Sizes of Cells



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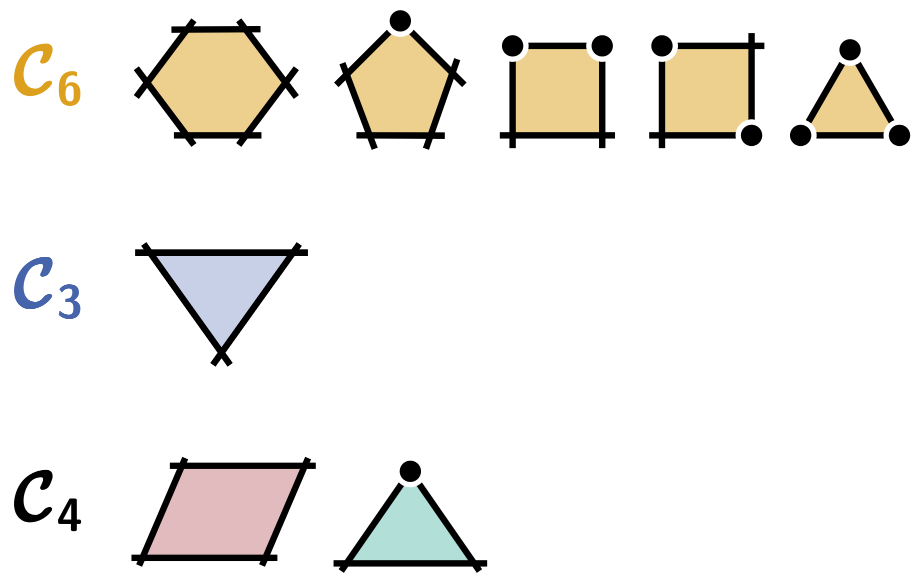
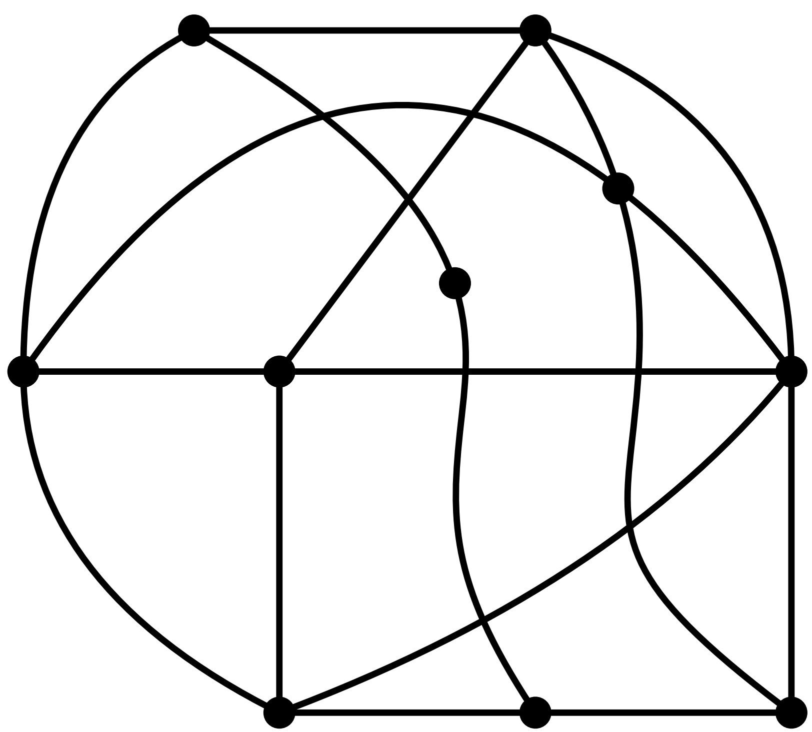


Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i\right) |\mathcal{C}_i| - |E|$$

(Kaufmann, Klemz, Knorr, Reddy, Schröder, Ueckerdt 2024)



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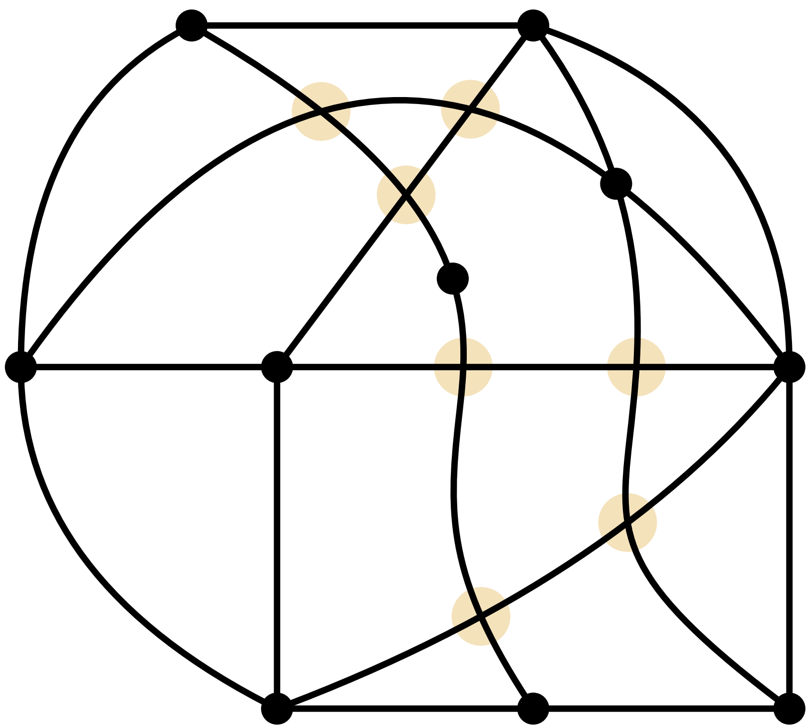

$7 =$

Density Formula

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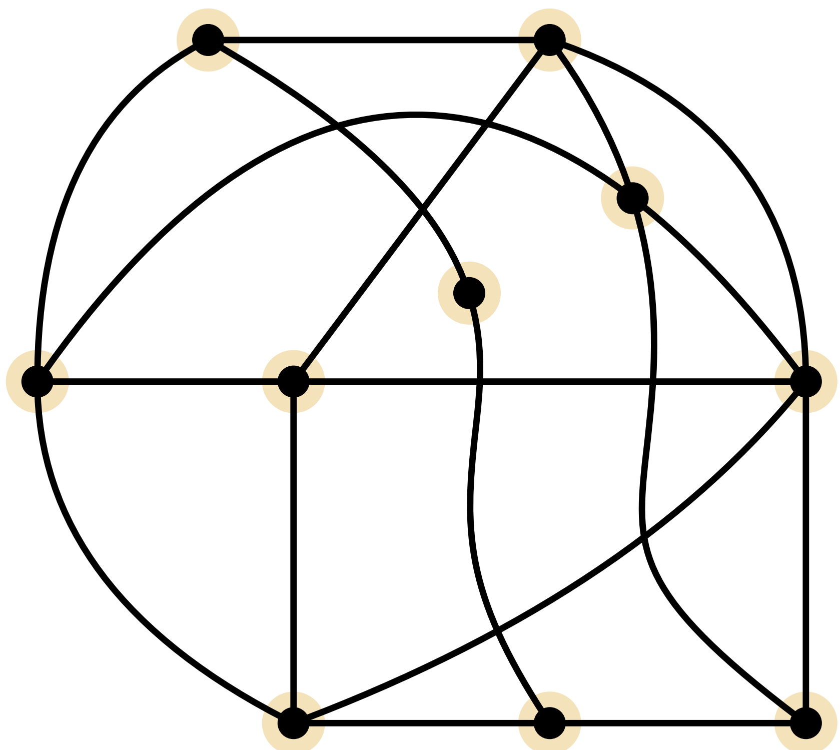

$$7 = t \cdot (10 - 2)$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

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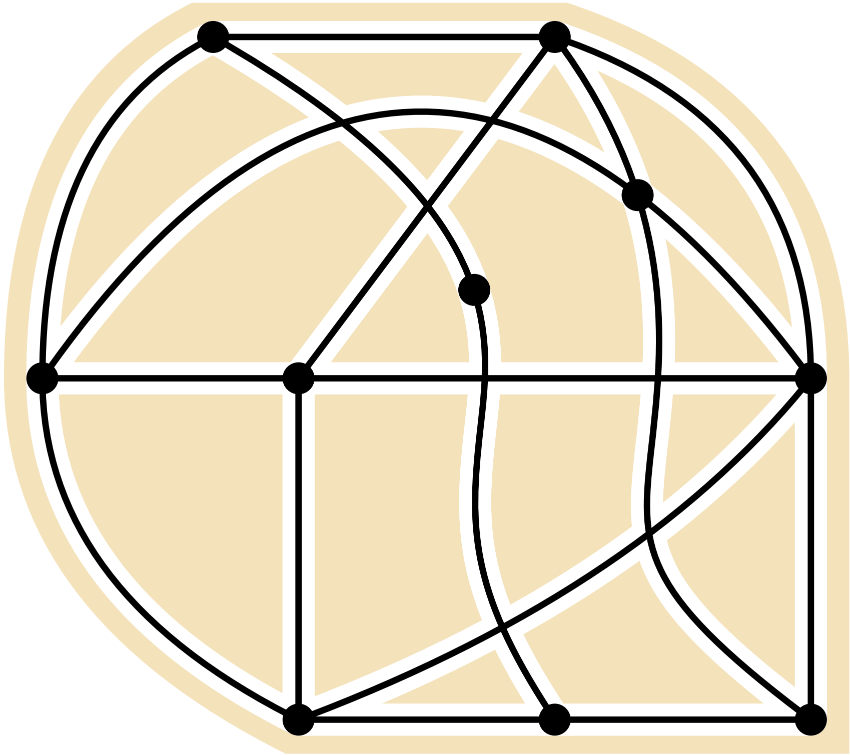

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i|$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

$$|X| = t \cdot (|V| - 2) + \sum_{i=1}^{\infty} \left(t + \frac{1-t}{4} i\right) |\mathcal{C}_i| - \boxed{|E|}$$

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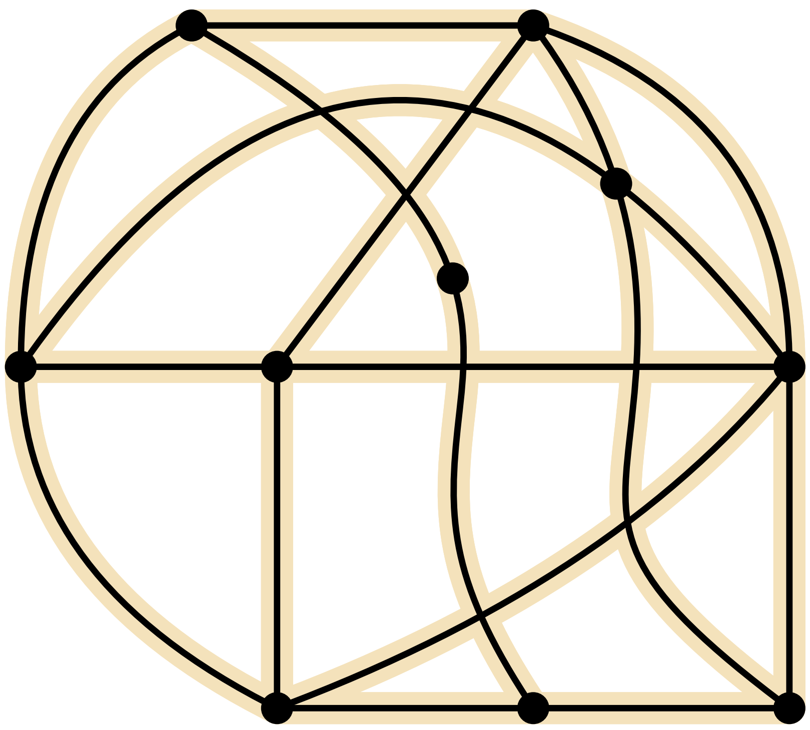

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - \boxed{18}$$

Density Formula

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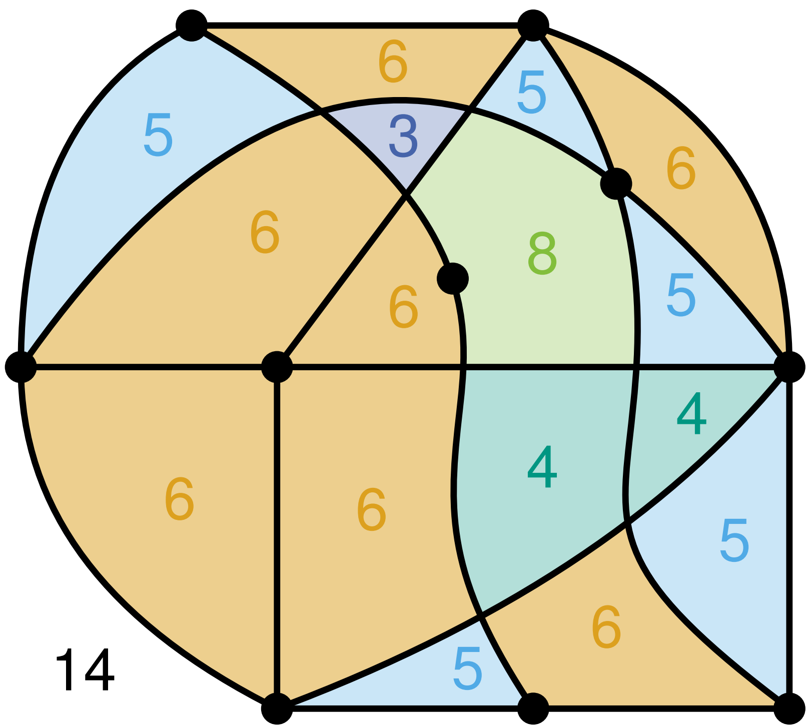

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

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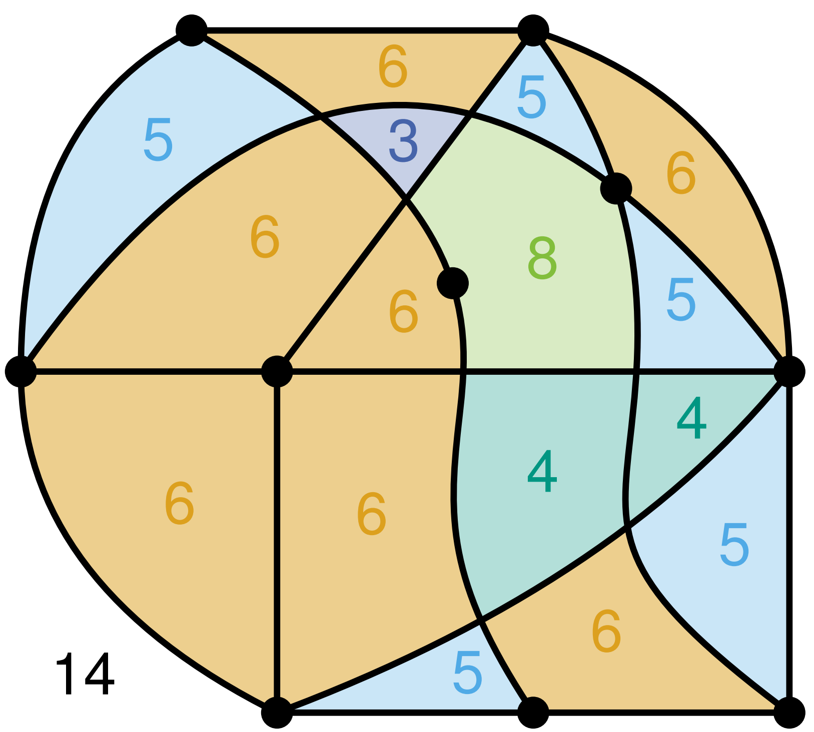

$t = 5$ $\quad |V| \geq 3$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i) |\mathcal{C}_i| - |E|$$

$$7 = t \cdot (10 - 2) + \sum_{i=1}^{\infty} w_{i,t} |\mathcal{C}_i| - 18$$

Density Formula

Lemma: For every $t \in \mathbb{R}$ and every connected drawing of a graph $G = (V, E)$ with $|E| \geq 1$, we have

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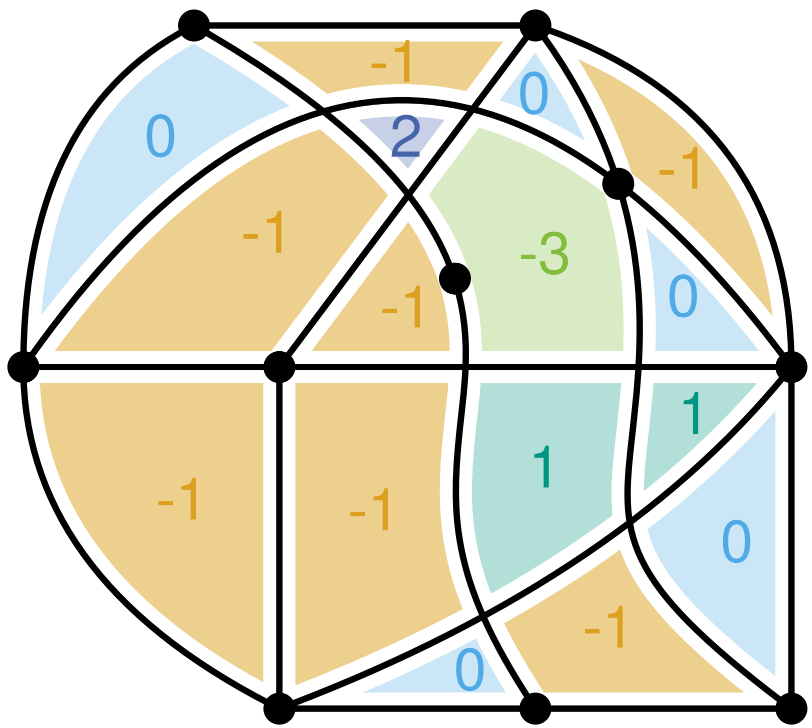

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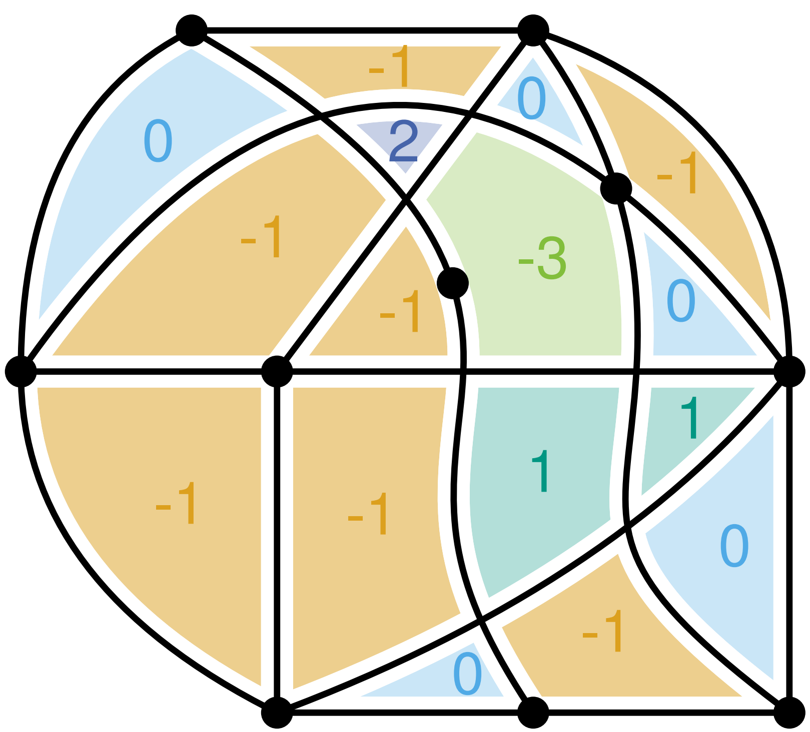
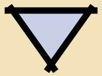
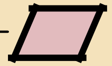


$t = 5$ $\quad |V| \geq 3$

$$|X| = 5(|V| - 2) + \sum_{i=3}^{\infty} (5 - i)\, |\mathcal{C}_i| - |E|$$

$|X| = 5(|V| - 2)$ $+2$  $+$  $+$ 

$$- \sum_{i=6}^{\infty} (i - 5)|\mathcal{C}_i| - |E|$$

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

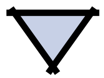
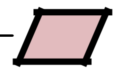
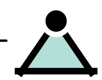
Obs: May assume no planar edges.

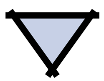
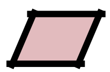
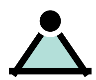
$$|X| = 5(n-2) + 2\bigtriangledown + \square + \bigtriangleup - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$$

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

$|X| = 5(n-2) + 2$  $+$  $+$  $- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m$

Idea: bound , ,  in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

$$|X| = 5(n-2) + 2\,\triangledown + \varparallelogram + \triangle - \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i| - m \qquad (\star)$$

Idea: bound $\triangledown$, $\varparallelogram$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$$\left\{\begin{array}{l} |X| \geq 3\,\triangledown \\ |X| \geq \varparallelogram \\ |X| \geq \triangle \\ |X| \leq m \\ (\star) \end{array}\right.$$

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

$$|X| = 5(n-2) + 2\,\bigtriangledown + \square + \dot{\triangle} \overbrace{- \textstyle\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}^{\leq 0} - m \qquad (\star)$$

Idea: bound $\bigtriangledown$, $\square$, $\dot{\triangle}$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$$\begin{cases} |X| \geq 3\,\bigtriangledown \\ |X| \geq \square \\ |X| \geq \dot{\triangle} \\ |X| \leq m \\ (\star) \end{cases}$$

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

$$|X| = 5(n-2) + 2 \quad + \quad + \quad \underbrace{- \sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound , , in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

LP
$$\begin{cases} |X| \geq 3 \\ |X| \geq \\ |X| \geq \\ |X| \leq m \\ (\star) \end{cases}$$

Maximize $|X|$ in LP

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

$$|X| = 5(n-2) + 2\,\triangledown + \square + \triangle \underbrace{- \textstyle\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

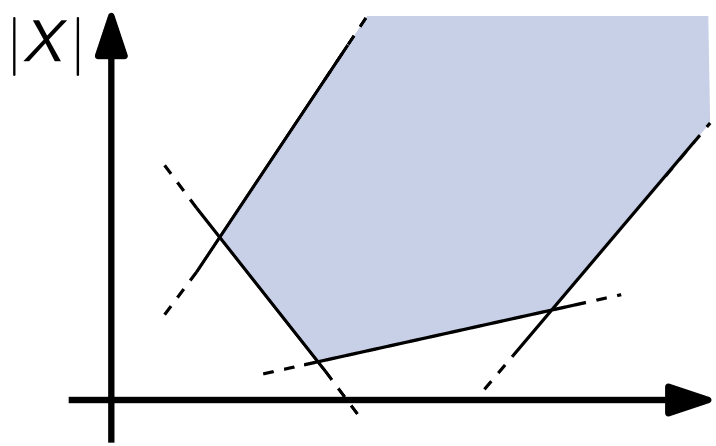
Idea: bound $\triangledown$, $\square$, $\triangle$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

LP $\begin{cases} |X| \geq 3\,\triangledown \\ |X| \geq \square \\ |X| \geq \triangle \\ |X| \leq m \\ (\star) \end{cases}$

Maximize $|X|$ in LP

$|X|$ unbounded



Karlsruher Institut für Technologie

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

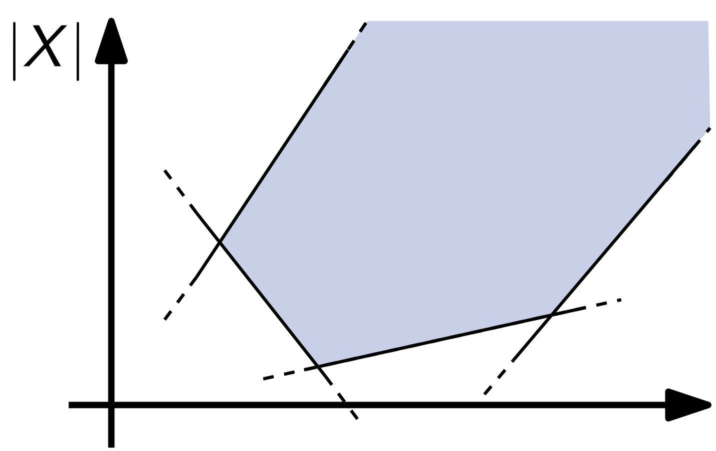
$$|X| = 5(n-2) + 2 \quad + \quad + \quad \underbrace{-\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound , , in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

$$\text{LP} \begin{cases} |X| \geq 3 \\ |X| \geq \\ |X| \geq \\ |X| \leq m \\ (\star) \end{cases}$$

Maximize $|X|$ in LP



$|X|$ unbounded

Idea: Need to take cells of size ≥ 6 into account

First Attempt

Question: Is $|X| \leq 3.\overline{3}n$ for every 2-plane drawing?

Obs: May assume no planar edges.

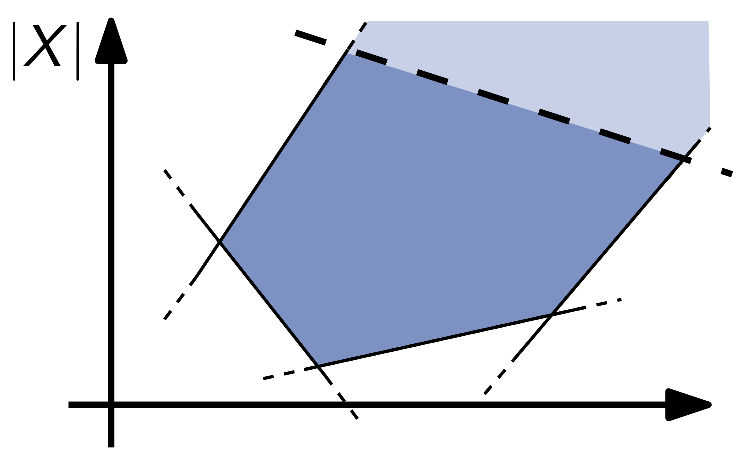
$$|X| = 5(n-2) + 2\bigtriangledown + \square + \bigtriangleup \underbrace{- \textstyle\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|}_{\leq 0} - m \qquad (\star)$$

Idea: bound $\bigtriangledown$, $\square$, $\bigtriangleup$ in terms of $|X|$, m and $\sum_{i=6}^{\infty}(i-5)|\mathcal{C}_i|$

Observation:

LP $\begin{cases} |X| \geq 3\bigtriangledown \\ |X| \geq \square \\ |X| \geq \bigtriangleup \\ |X| \leq m \\ (\star) \end{cases}$

Maximize $|X|$ in LP



$|X|$ unbounded

Idea: Need to take cells of size ≥ 6 into account

Large Cells

Question: Number of large cells?

Large Cells

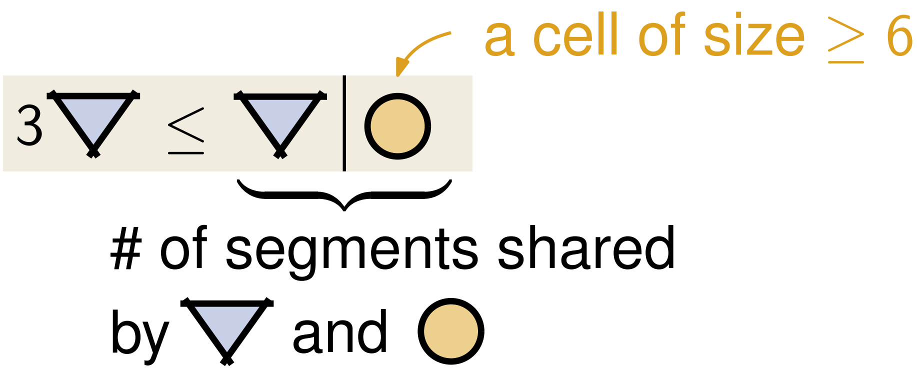
Question: Number of large cells?

a cell of size ≥ 6

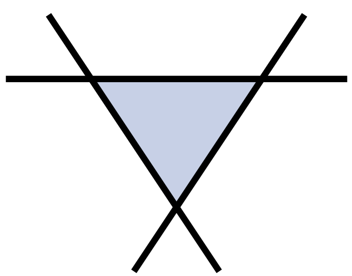
$$3\,\triangledown \leq \underbrace{\triangledown \mid \bigcirc}_{\text{\# of segments shared by } \triangledown \text{ and } \bigcirc}$$

Large Cells

Question: Number of large cells?



Proof:



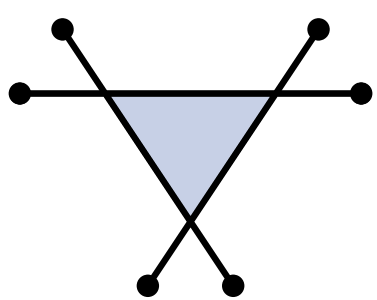
Large Cells

Question: Number of large cells?

a cell of size ≥ 6

$$3\,\triangledown \leq \underbrace{\triangledown \mid \bigcirc}_{\text{\# of segments shared by } \triangledown \text{ and } \bigcirc}$$

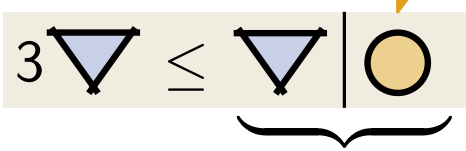
Proof:



Large Cells

Question: Number of large cells?

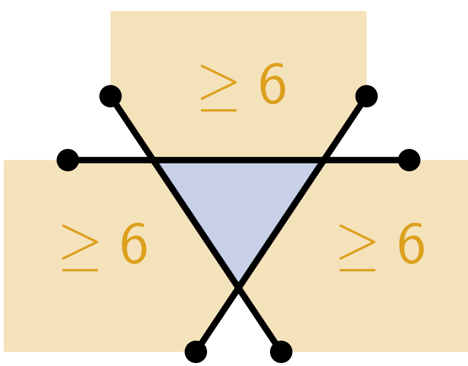
a cell of size ≥ 6



# of segments shared

by ▽ and ○

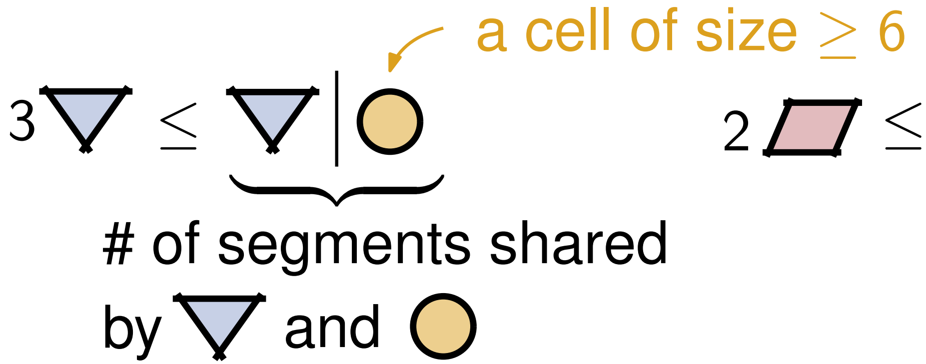
Proof:

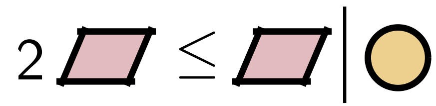


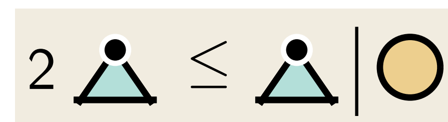
Large Cells

Question: Number of large cells?

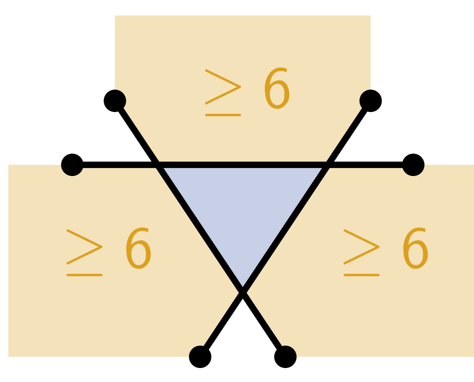
Recall: May assume no planar edges.

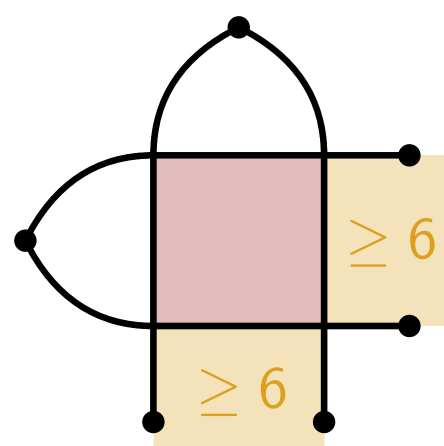


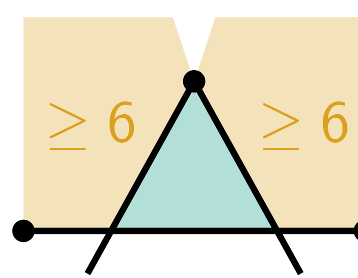




Proof:







Large Cells

Question: Sizes of large cells?

Large Cells

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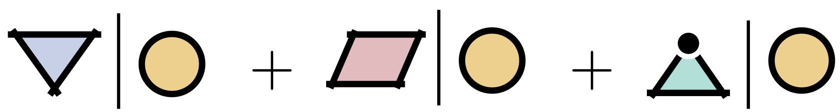
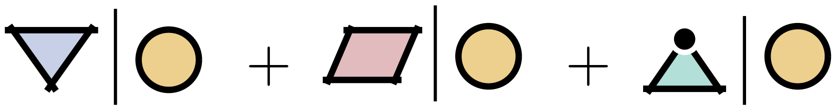

$\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

Large Cells

Question: Sizes of large cells?



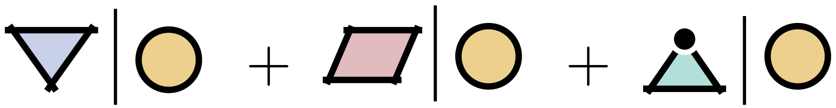
$\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

$\geq$ # segments

Idea: count # of segments on large cells

Large Cells

Question: Sizes of large cells?

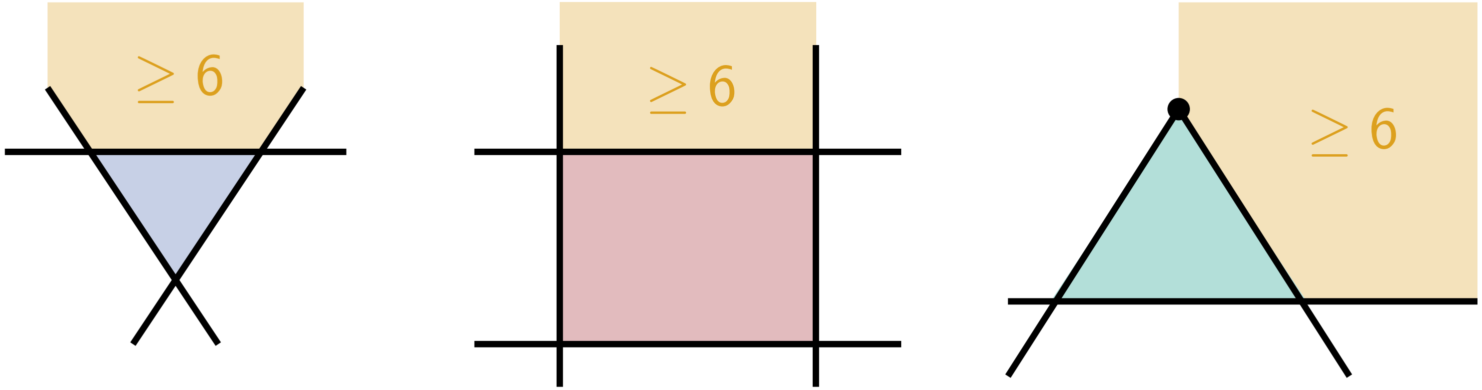


$\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

$\geq$ # segments

Idea: count # of segments on large cells

Proof:



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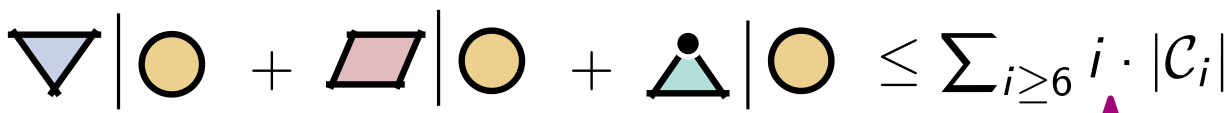

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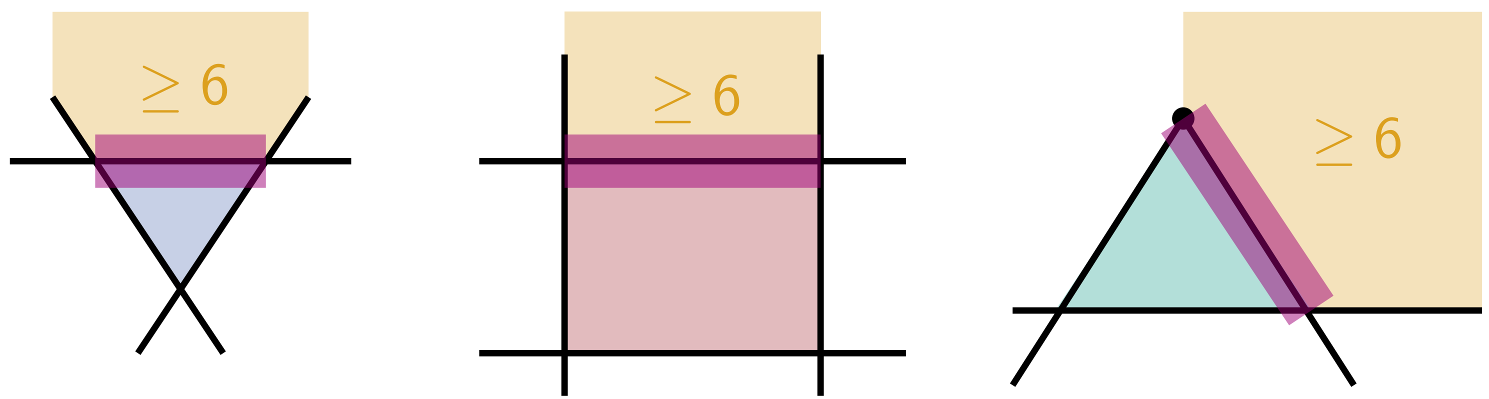

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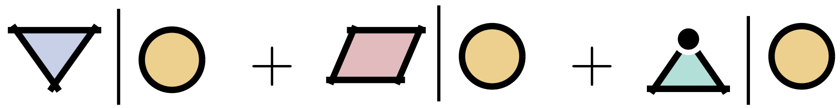 $\leq \sum_{i\geq 6} i \cdot |\mathcal{C}_i|$

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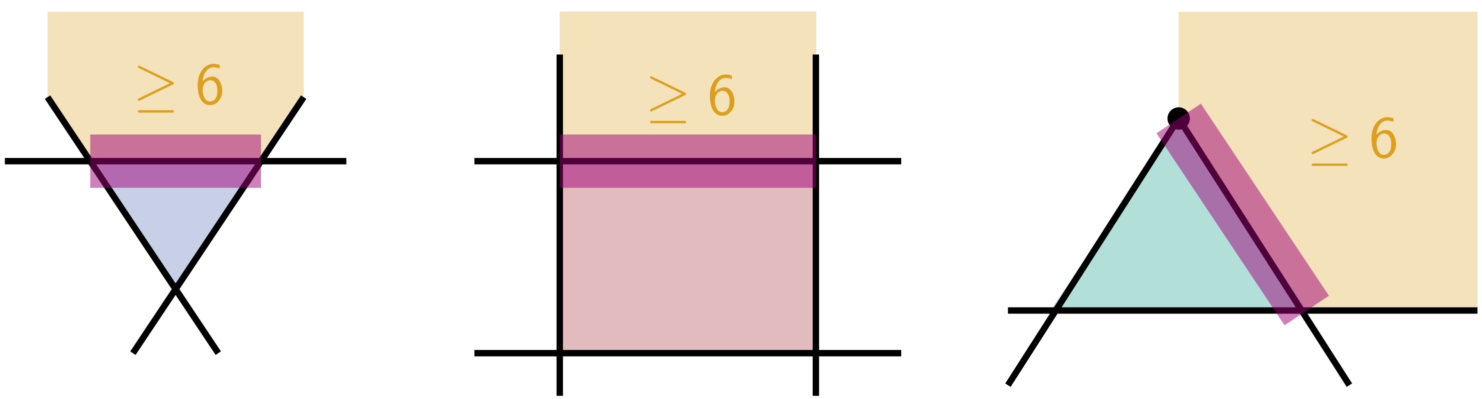

Obs: $\frac{1}{6}\sum_{i\geq 6} i \cdot |\mathcal{C}_i| \leq \sum_{i\geq 6}(i-5)|\mathcal{C}_i|$

Large Cells

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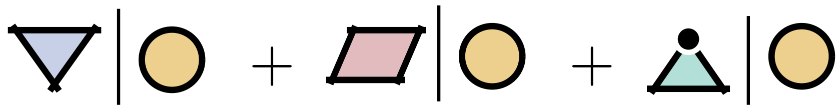 $\leq \sum_{i \geq 6} i \cdot |\mathcal{C}_i|$

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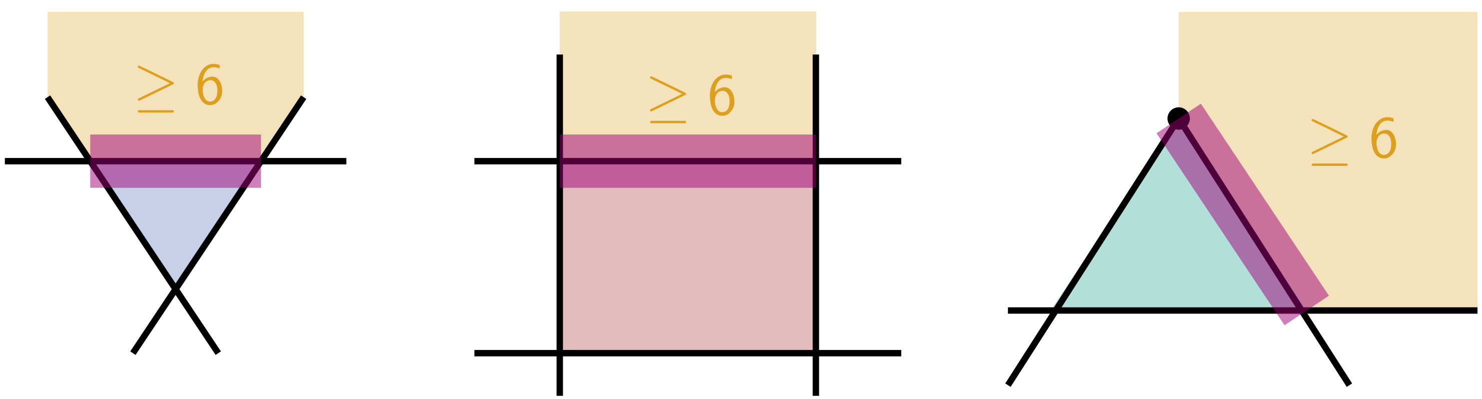
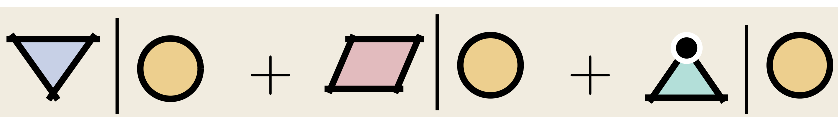
Proof:

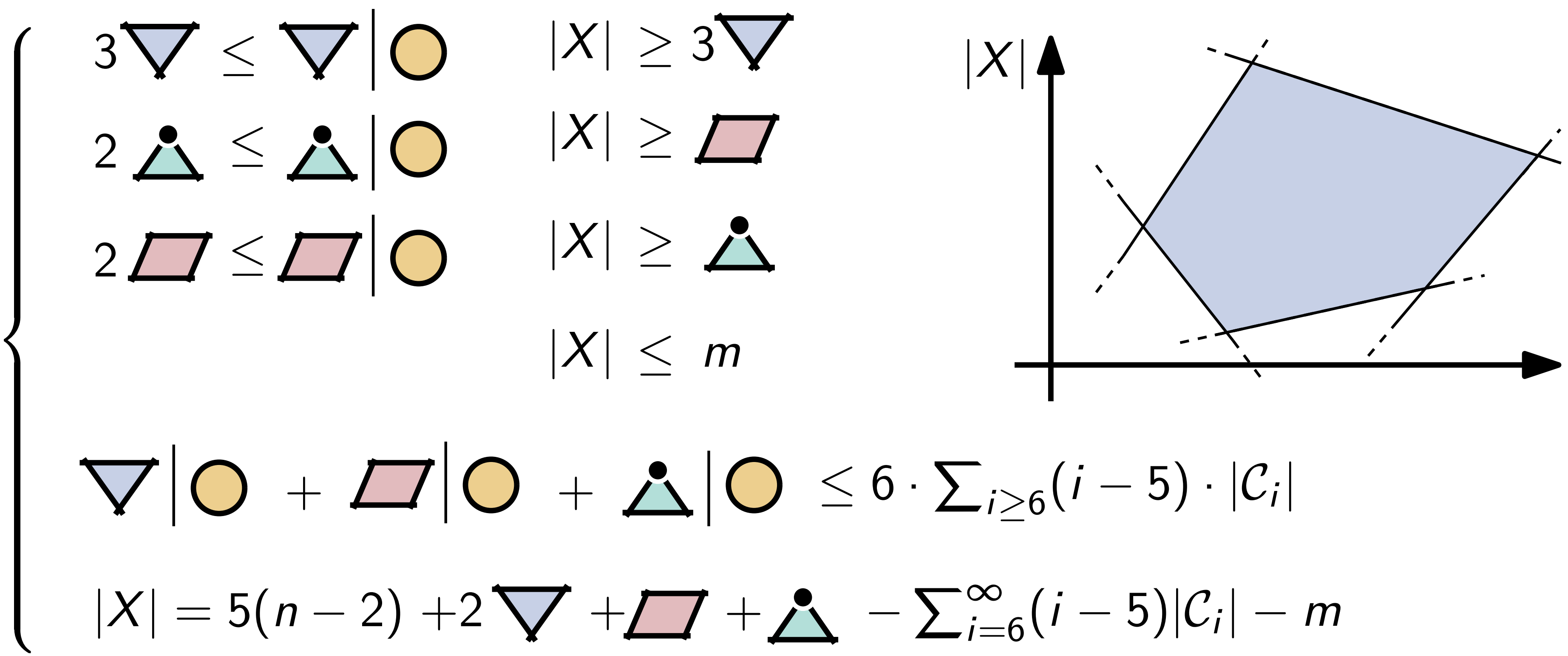


Obs: $\frac{1}{6} \sum_{i \geq 6} i \cdot |\mathcal{C}_i| \leq \sum_{i \geq 6} (i-5) |\mathcal{C}_i|$

$\rightsquigarrow$  $\leq 6 \cdot \sum_{i \geq 6} (i-5) \cdot |\mathcal{C}_i|$

Second Attempt

Inequations: **Visualization:**



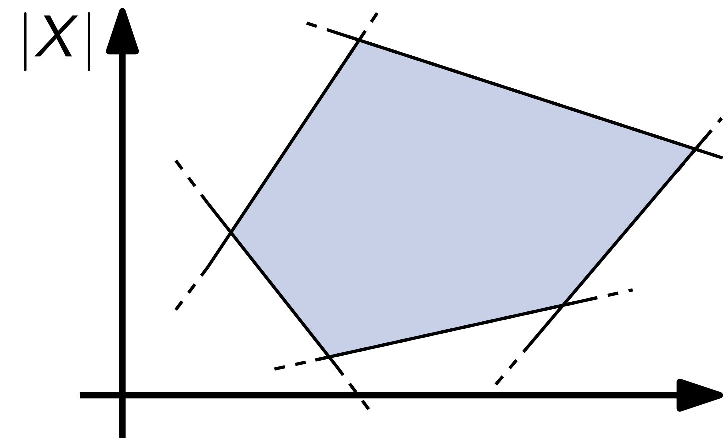
Maximize $|X|$ in LP

Second Attempt

Inequations:

$$
\begin{cases}
3\,\bigtriangledown \le \bigtriangledown \,\Big|\, \bigcirc \qquad & |X| \ge 3\,\bigtriangledown \\
2\,\dot{\bigtriangleup} \le \dot{\bigtriangleup} \,\Big|\, \bigcirc & |X| \ge \varparallelogram \\
2\,\varparallelogram \le \varparallelogram \,\Big|\, \bigcirc & |X| \ge \dot{\bigtriangleup} \\
 & |X| \le m \\
\bigtriangledown \,\Big|\, \bigcirc + \varparallelogram \,\Big|\, \bigcirc + \dot{\bigtriangleup} \,\Big|\, \bigcirc \le 6 \cdot \sum_{i \ge 6} (i-5) \cdot |\mathcal{C}_i| & \\
|X| = 5(n-2) + 2\,\bigtriangledown + \varparallelogram + \dot{\bigtriangleup} - \sum_{i=6}^{\infty} (i-5)|\mathcal{C}_i| - m &
\end{cases}
$$

Visualization:



Maximize $|X|$ in LP

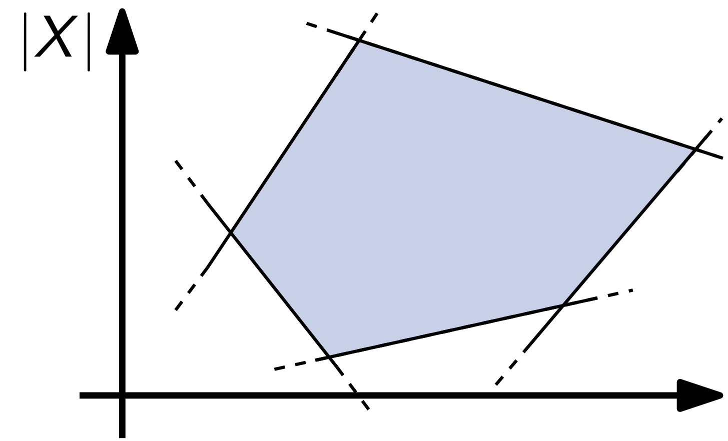
Want: $|X| \le 3.\overline{3}n$

Second Attempt

Inequations:

$$
\begin{cases}
3\triangledown \le \triangledown | \bigcirc & |X| \ge 3\triangledown \\
2\dot\triangle \le \dot\triangle | \bigcirc & |X| \ge \square \\
2\square \le \square | \bigcirc & |X| \ge \dot\triangle \\
 & |X| \le m \\
\triangledown | \bigcirc + \square | \bigcirc + \dot\triangle | \bigcirc \le 6 \cdot \sum_{i \ge 6} (i-5) \cdot |\mathcal{C}_i| & \\
|X| = 5(n-2) + 2\triangledown + \square + \dot\triangle - \sum_{i=6}^{\infty} (i-5)|\mathcal{C}_i| - m &
\end{cases}
$$

Visualization:



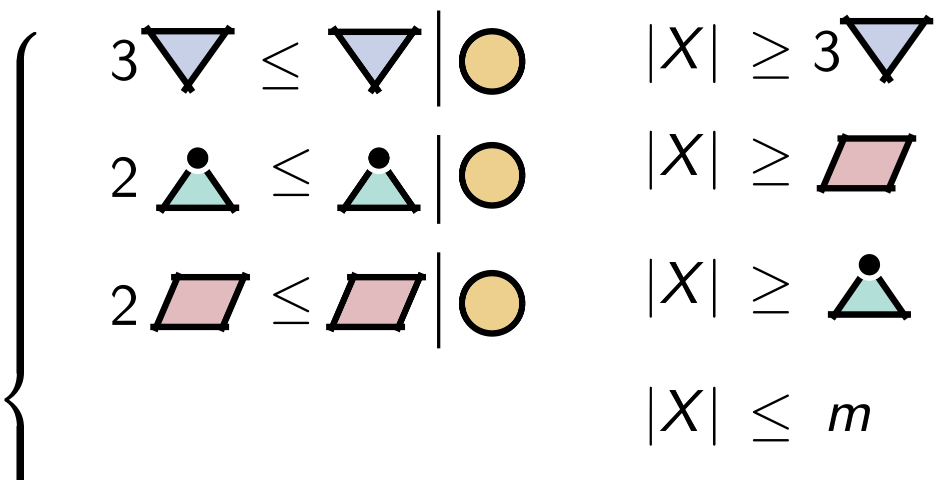
$\rightsquigarrow$ **Idea:** Maximize $|X|$ in LP

Obtain: $|X| \le 30n$

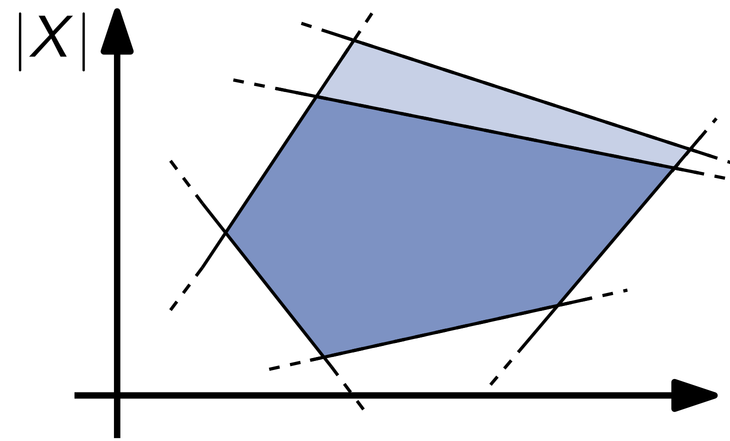
Want: $|X| \le 3.\overline{3}n$

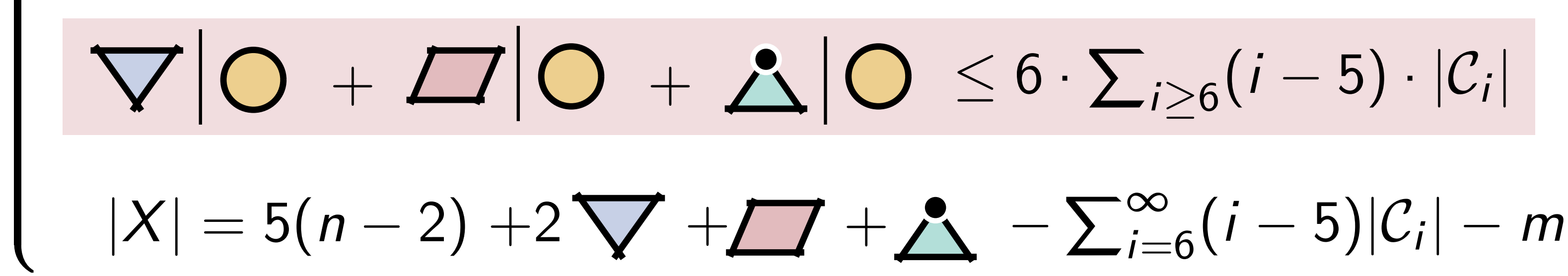
Second Attempt

Inequations:



Visualization:



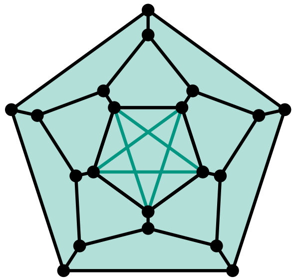


Idea: Maximize $|X|$ in LP — **Obtain:** $|X| \le \cancel{3n}$ $3.\overline{3}n$

Want: $|X| \le 3.\overline{3}n$ — Maximize $|E|$ in LP — **Obtain:** $|E| \le 5n$

Overview

Results



	2-plane
m	$\leq 5n$ (Pach, Tóth 1997)
$\lvert X \rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)

Overview

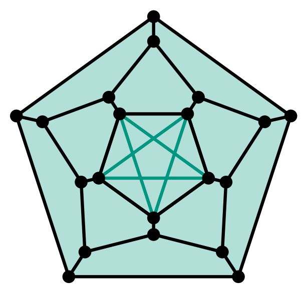
Karlsruher Institut für Technologie

Technique

LP: $2\,\square \leq \square \,|\, \bigcirc$; $2\,\square \leq \square \,|\, \bigcirc$; $\dots$; Density Formula

Maximize $|X|$ in LP $\rightsquigarrow$ Bound on $|X|$

Results



	2-plane
m	$\leq 5n$ (Pach, Tóth 1997)
$\lvert X\rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)

Overview

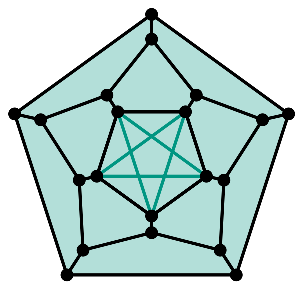
Technique

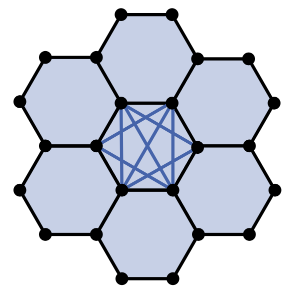
LP $\begin{cases} 2 \leq | \\ 2 \leq | \\ \dots \\ \text{Density Formula} \end{cases}$

Maximize $|X|$ in LP

Bound on $|X|$

Results





	2-plane	3-plane
m	$\leq 5n$ (Pach, Tóth 1997)	$\leq 5.5n$ (Pach, Radoicic, Tardos, Tóth 2006)
$\lvert X\rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)	$\leq 5.5n$

Overview

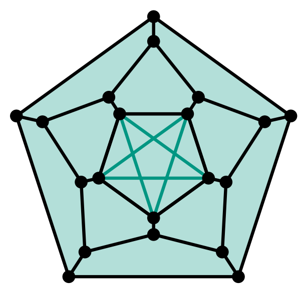
Technique

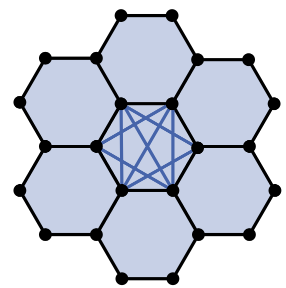
LP
$2 \leq |$
$2 \leq |$
. . .
Density Formula

Maximize $|X|$ in LP

Bound on $|X|$

Results





	2-plane	3-plane	4-plane
m	$\leq 5n$ (Pach, Tóth 1997)	$\leq 5.5n$ (Pach, Radoicic, Tardos, Tóth 2006)	$\leq 6n$ (Ackerman 2015)
$\lvert X\rvert$	$\leq 3.\overline{3}n$ (Bekos et al. 2024)	$\leq 5.5n$	?