Algorithms for graph visualization

Contact representations of planar graphs.
In a **contact representation** of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.
Contact representation

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![Contact representation diagram]
Contact representation

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- 6-gons are necessary and sufficient for planar graphs! (Gansner et. al. 2010)
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.

Every 3-connected cubic planar graph admits a contact representation with triangles (Kobourov et. al. 2012)
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.

Every 4-connected planar graph admits a contact representation with rectangles (Xin He 1993)
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.

- Every triangle-free planar graph has a contact representation with line segments in just three directions (de Castro et. al. 1999)
In a contact representation of a planar graph each vertex is represented as a geometrical object such that two object touch if and only if the corresponding vertices are connected by an edge.

Each planar graph has a touching disks representation (Koebe 1936)
Application: visualization of a clustering
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- Run force directed algorithm on the graph
- Compute voronoi diagram of the points representing the vertices
Application: visualization of a clustering

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- Color voronoi cell according to the clustering
Application: visualization of a clustering

- Run force directed algorithm on the graph
- Compute voronoi diagram of the points representing the vertices
- Color voronoi cell according to the clustering
- Merge cells of the same cluster

Inspired by GMap (Gansner et al.)
Application: visualization of a clustering

- Just a heuristic, there are no guarantees that:
  - The strong adjacencies between clusters are represented by contacts of the countries
Application: visualization of a clustering

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  - The length of the boundary is representing the strength of the adjacency
Application: visualization of a clustering

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Application: visualization of a clustering

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Application: visualization of a clustering

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In a contact representation of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.

Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)
Today

Contact representation

In a contact representation of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.

Contact representation with rectangles

- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)
- A contact representation of $G$ with rectangles, without holes and with rectangular outer boundary is called a rectangular dual of $G$
Rectangular Dual

Which graphs have a rectangular dual?
Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

Let $G$ be a graph. A triangle $C$ of $G$ whose removal results in at least two disconnected components is called a **separating triangle** of $G$. 
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Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)
Rectangular Dual

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Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)

No four rectangles meet a a point! Each face of $G$ must be a triangle!

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Rectangular Dual

Necessary conditions for a planar graph $G$ to have a rectangular dual:

- $G$ must have at least 4 vertices on the outer face
- $G$ must have no separating triangle
- each internal face of $G$ must be a triangle
Rectangular Dual

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We will prove that these conditions are sufficient!
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We will prove that these conditions are sufficient!


A planar graph $G = (V, E)$ has a rectangular dual $R$ with four rectangles on the boundary of $R$ if and only if the following conditions hold:

- Every interior face of $G$ is a triangle and the exterior face of $G$ is a quadrangle;
- $G$ has no separating triangles
Rectangular Dual

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Proper Triangular Planar Graph (PTP)
In order to construct a rectangular dual we need to partition our edges on **vertical** and **horizontal**. **Regular edge labeling** (REL, for short) is a tool for that.
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Regular edge labeling

For each internal vertex:

For the boundary vertices:

\[ vW \quad vN \quad vS \quad vE \]
Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of $G$ $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph $G_{k-1}$ induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary $C_{k-1}$ of $G_{k-1}$ contains the edge $(v_S, v_W)$.
- $v_k$ is in exterior face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k - 2$, $v_k$ has at least 2 neighbors in $G \setminus G_{k-1}$.
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Canonical ordering with extra condition on $v_k$!
Rectangular Dual

Theorem (Refined canonical ordering)

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of $G$ $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph $G_{k-1}$ induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary $C_{k-1}$ of $G_{k-1}$ contains the edge $(v_S, v_W)$.
- $v_k$ is in exterior face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k - 2$, $v_k$ has at least 2 neighbors in $G \setminus G_{k-1}$.

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Rectangular Dual

Refined canonical ordering
Rectangular Dual

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9 - 5
Rectangular Dual

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Rectangular Dual

Given a refined canonical ordering of $G$ we construct a REL as follows:

- For each $(v_i, v_j)$ orient it from $v_i$ to $v_j$, for $i < j$;
- Base edge of $v_k$ is $(v_l, v_k)$, where $l < k$ is minimal.
- $v_k$ has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that $v_{t_1}$ is left point of $v_k$ and $v_{t_l}$ is right point of $v_k$.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of $v_k$, we call $(v_k, v_{k_1})$ left edge and $(v_k, v_{k_l})$ right edge.
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**Lemma 1**

Left edge or right edge can not be a base edge.
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Proof: Assume that left edge $(v_k, v_{k_1})$ is the base edge of $v_{k_1}$.
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**Lemma 2**

An edge is either a left edge, a right edge or a base edge.

**Proof:**

- The exclusive “or” follows from Lemma 1.
- Let $(v_{t_a}, v_k)$ be base edge of $v_k$.
- $v_{t_a}$ is right point of $v_{t_{a-1}}, v_{t_{a-1}}$ is right point of $v_{t_{a-2}}, \text{generally } v_{t_{i+1}}$ is right point of $v_{t_i}, 1 \leq i < a - 1$
- Edges $(v_{t_i}, v_k), 1 \leq i < a - 1$, are right edges;
- Similarly we prove that edges $(v_{t_i}, v_k), a + 1 \leq i < l$, are left edges;
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- Edges $(v_{t_i}, v_k)$, $1 \leq i < a - 1$, are right edges;
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Rectangular Dual

right edges

left edges

base edge

$v_k$
Rectangular Dual

right edges

base edge

left edges

$v_k$

$u$
Rectangular Dual
Rectangular Dual
We call $T_b$ blue edges and $T_r$ red edges.
Rectangular Dual

We call $T_b$ blue edges and $T_r$ red edges.

**Lemma 3**

$\{T_r, T_b\}$ is a regular edge labeling.

**Proof:**

$k_l \geq 2$
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**Proof:**
Rectangular Dual

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Lemma 3
{$T_r, T_b$} is a regular edge labeling.

Proof:

$$k_d = \max\{v_k_1 \cdots v_k_l\}$$

$$k_l \geq 2$$
Rectangular Dual

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**Proof:**

$k_d = \max\{v_{k_1} \ldots v_{k_l}\}$

$k_1 < k_2 < \ldots < k_d \text{ and } k_d > k_{d+1} > \ldots > k_l$

$v_{k_1}$

$\{v_{k_2}, \ldots v_{k_{l-1}}\}$

$v_{k_l} \geq 2$
Rectangular Dual

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\[ k_d = \max\{v_{k_1} \cdots v_{k_l}\} \quad \Rightarrow \quad k_1 < k_2 < \cdots < k_d \quad \text{and} \quad k_d > k_{d+1} > \cdots > k_l \]
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$k_d = \max\{v_{k_1} \cdots v_{k_l}\}$

$k_1 < k_2 < \cdots < k_d$ and

$k_d > k_{d+1} > \cdots > k_l$

$(v_k, v_{k_i}), 2 \leq i \leq d - 1$ are red

$(v_k, v_{k_i}), d + 1 \leq i \leq l - 1$ are blue

edge $(v_k, v_{k_d})$ is either red or blue
Rectangular Dual

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Rectangular Dual

Algorithmen zur Visualisierung von Graphen
Tamara Mchedlidze

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Lehrstuhl Algorithmik I
Rectangular Dual
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

W-E net $G_{W-E}$
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Algorithmen zur Visualisierung von Graphen

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Rectangular Dual
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Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a S-N net $G_{S-N}$ of $G$ (consists of $T_r$ plus outer edges)
- Construct the dual $G^*_{S-N}$ of $G_{S-N}$ and compute a topological ordering $f_{sn}$ of $G^*_{S-N}$
- For each vertex $v \in V$, let $f$ and $g$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$
### Rectangular Dual

**Algorithm Rectangular dual**

**Input:** A PTP graph $G = (V, E)$

1. Find a REL $T_r, T_b$ of $G$
2. Construct a S-N net $G_{S-N}$ of $G$ (consists of $T_r$ plus outer edges)
3. Construct the dual $G^{\star}_{S-N}$ of $G_{S-N}$ and compute a topological ordering $f_{sn}$ of $G^{\star}_{S-N}$
4. For each vertex $v \in V$, let $f$ and $g$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
5. Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$
Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

1. Find a REL $T_r, T_b$ of $G$;
2. Construct a W-E net $G_{W-E}$ of $G$ (consists of $T_b$ plus outer edges);
3. Construct the dual $G_{SN}$ and compute a topological ordering $f_{SN}$ of $G_{SN}$;
4. For each vertex $v \in V$, let $f$ and $g$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{SN}(f)$ and $x_2(v) = f_{SN}(g)$.
5. Define $x_1(v_{SN}) = x_1(v_{S}) = 1$ and $x_2(v_{SN}) = x_2(v_{S}) = max f_{SN} - 1$. 
Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

1. Find a REL $T_r, T_b$ of $G$;
2. Construct a W-E net $GW - E$ of $G$ (consists of $T_b$ plus outer edges);
3. Construct the dual $G^*_{W - E}$ and compute a topological ordering $f_{we}$ of $G^*_{W - E}$;
4. For each vertex $v \in V$, let $f$ and $g$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
5. Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = max f_{sn} - 1$.
Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a $W-E$ net of $G$ (consists of $T_b$ plus outer edges);
- Construct the dual $G^*_{W-E}$ of $G$ and compute a topological ordering $f_{we}$ of $G^*_{W-E}$;
- For each vertex $v \in V$, let $f$ and $g$ be the face below and face above $v$. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$;
- Define $x_1(v) = 1$ and $x_2(v) = \max f_{sn} - 1$.
Rectangular Dual

Algorithm Rectangular dual

Input: A PTP graph \( G = (V, E) \)

- Find a REL \( T_r, T_b \) of \( G \);
- Construct a W-E net \( G_{W-E} \) of \( G \) (consists of \( T_b \) plus outer edges)
- Construct the dual \( G^*_{W-E} \) of \( G_{W-E} \) and compute a topological ordering \( f_{we} \) of \( G^*_{W-E} \)
- For each vertex \( v \in V \), let \( f \) and \( g \) be the face below and face above \( v \). Set \( y_1(v) = f_{sn}(f) \) and \( y_2(v) = f_{sn}(g) \).
- Define \( y_1(v_W) = y_1(s_E) = 0 \) and \( y_1(v_W) = y_1(s_E) = \max f_{we} \)
Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a W-E net of $G$ (consists of $T_b$ plus outer edges);
- Construct the dual of $G^*$ and compute a topological ordering $f_{we}$ of $G^*_{W-E}$;
- For each vertex $v \in V$, let $f$ and $g$ be the face below and face above $v$. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$.
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v), x_2(v)$ and y-coordinates $y_1(v), y_2(v)$.
Rectangular Dual
Rectangular Dual

$x_1(v_N) = 1, \quad x_2(v_N) = 15$
$x_1(v_S) = 1, \quad x_2(v_S) = 15$
$x_1(v_W) = 0, \quad x_2(v_W) = 1$
$x_1(v_E) = 15, \quad x_2(v_E) = 16$
$x_1(a) = 1, \quad x_2(a) = 3$
$x_1(b) = 3, \quad x_2(b) = 5$
$x_1(c) = 5, \quad x_2(c) = 14$
$x_1(d) = 14, \quad x_2(d) = 15$
$x_1(e) = 13, \quad x_2(e) = 15$
Rectangular Dual

\[
x_1(v_N) = 1, \quad x_2(v_N) = 15 \\
x_1(v_S) = 1, \quad x_2(v_S) = 15 \\
x_1(v_W) = 0, \quad x_2(v_W) = 1 \\
x_1(v_E) = 15, \quad x_2(v_E) = 16 \\
x_1(a) = 1, \quad x_2(a) = 3 \\
x_1(b) = 3, \quad x_2(b) = 5 \\
x_1(c) = 5, \quad x_2(c) = 14 \\
x_1(d) = 14, \quad x_2(d) = 15 \\
x_1(e) = 13, \quad x_2(e) = 15
\]
Rectangular Dual

\[ x_1(v_N) = 1, \ x_2(v_N) = 15 \]
\[ x_1(v_S) = 1, \ x_2(v_S) = 15 \]
\[ x_1(v_W) = 0, \ x_2(v_W) = 1 \]
\[ x_1(v_E) = 15, \ x_2(v_E) = 16 \]
\[ x_1(a) = 1, \ x_2(a) = 3 \]
\[ x_1(b) = 3, \ x_2(b) = 5 \]
\[ x_1(c) = 5, \ x_2(c) = 14 \]
\[ x_1(d) = 14, \ x_2(d) = 15 \]
\[ x_1(e) = 13, \ x_2(e) = 15 \]
Rectangular Dual

\[ x_1(v_N) = 1, \quad x_2(v_N) = 15 \]
\[ x_1(v_S) = 1, \quad x_2(v_S) = 15 \]
\[ x_1(v_W) = 0, \quad x_2(v_W) = 1 \]
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\[ x_1(a) = 1, \quad x_2(a) = 3 \]
\[ x_1(b) = 3, \quad x_2(b) = 5 \]
\[ x_1(c) = 5, \quad x_2(c) = 14 \]
\[ x_1(d) = 14, \quad x_2(d) = 15 \]
\[ x_1(e) = 13, \quad x_2(e) = 15 \]
Rectangular Dual

$x_1(v_N) = 1$, $x_2(v_N) = 15$
$x_1(v_S) = 1$, $x_2(v_S) = 15$
$x_1(v_W) = 0$, $x_2(v_W) = 1$
$x_1(v_E) = 15$, $x_2(v_E) = 16$
$x_1(a) = 1$, $x_2(a) = 3$
$x_1(b) = 3$, $x_2(b) = 5$
$x_1(c) = 5$, $x_2(c) = 14$
$x_1(d) = 14$, $x_2(d) = 15$
$x_1(e) = 13$, $x_2(e) = 15$
Rectangular Dual

\[ x_1(v_N) = 1, \quad x_2(v_N) = 15 \]
\[ x_1(v_S) = 1, \quad x_2(v_S) = 15 \]
\[ x_1(v_W) = 0, \quad x_2(v_W) = 1 \]
\[ x_1(v_E) = 15, \quad x_2(v_E) = 16 \]
\[ x_1(a) = 1, \quad x_2(a) = 3 \]
\[ x_1(b) = 3, \quad x_2(b) = 5 \]
\[ x_1(c) = 5, \quad x_2(c) = 14 \]
\[ x_1(d) = 14, \quad x_2(d) = 15 \]
\[ x_1(e) = 13, \quad x_2(e) = 15 \]

\[ y_1(v_W) = 0, \quad y_2(v_W) = 10 \]
\[ y_1(v_E) = 0, \quad y_2(v_E) = 10 \]
\[ y_1(v_N) = 9, \quad y_2(v_N) = 10 \]
\[ y_1(v_S) = 0, \quad y_2(v_S) = 1 \]
\[ y_1(a) = 1, \quad y_2(a) = 2 \]
\[ y_1(b) = 1, \quad y_2(b) = 2 \]
\[ y_1(c) = 1, \quad y_2(c) = 2 \]
\[ y_1(d) = 1, \quad y_2(d) = 2 \]
\[ y_1(e) = 2, \quad y_2(e) = 3 \]
Rectangular Dual

\[ x_1(v_N) = 1, \quad x_2(v_N) = 15 \]
\[ x_1(v_S) = 1, \quad x_2(v_S) = 15 \]
\[ x_1(v_W) = 0, \quad x_2(v_W) = 1 \]
\[ x_1(v_E) = 15, \quad x_2(v_E) = 16 \]
\[ x_1(a) = 1, \quad x_2(a) = 3 \]
\[ x_1(b) = 3, \quad x_2(b) = 5 \]
\[ x_1(c) = 5, \quad x_2(c) = 14 \]
\[ x_1(d) = 14, \quad x_2(d) = 15 \]
\[ x_1(e) = 13, \quad x_2(e) = 15 \]

\[ y_1(v_W) = 0, \quad y_2(v_W) = 10 \]
\[ y_1(v_E) = 0, \quad y_2(v_E) = 10 \]
\[ y_1(v_N) = 9, \quad y_2(v_N) = 10 \]
\[ y_1(v_S) = 0, \quad y_2(v_S) = 1 \]
\[ y_1(a) = 1, \quad y_2(a) = 2 \]
\[ y_1(b) = 1, \quad y_2(b) = 2 \]
\[ y_1(c) = 1, \quad y_2(c) = 2 \]
\[ y_1(d) = 1, \quad y_2(d) = 2 \]
\[ y_1(e) = 2, \quad y_2(e) = 3 \]
Rectangular Dual

$x_1(v_N) = 1$, $x_2(v_N) = 15$
$x_1(v_S) = 1$, $x_2(v_S) = 15$
$x_1(v_W) = 0$, $x_2(v_W) = 1$
$x_1(v_E) = 15$, $x_2(v_E) = 16$
$x_1(a) = 1$, $x_2(a) = 3$
$x_1(b) = 3$, $x_2(b) = 5$
$x_1(c) = 5$, $x_2(c) = 14$
$x_1(d) = 14$, $x_2(d) = 15$
$x_1(e) = 13$, $x_2(e) = 15$

$y_1(v_W) = 0$, $y_2(v_W) = 10$
$y_1(v_E) = 0$, $y_2(v_E) = 10$
$y_1(v_N) = 9$, $y_2(v_N) = 10$
$y_1(v_S) = 0$, $y_2(v_S) = 1$
$y_1(a) = 1$, $y_2(a) = 2$
$y_1(b) = 1$, $y_2(b) = 2$
$y_1(c) = 1$, $y_2(c) = 2$
$y_1(d) = 1$, $y_2(d) = 2$
$y_1(e) = 2$, $y_2(e) = 3$
Rectangular Dual

In the following we prove that presented algorithm constructs a rectangular dual of $G$.


- Let $f_1, \ldots, f_k$ be the faces of $G^*_{S-N}$ (resp. $G^*_{W-E}$), enumerated according to $st$-numbering $f_{sn}$ (resp. $f_{we}$).

- Let $G^i_{S-N}$ (resp. $G^i_{W-E}$) denote the subgraph of $G$ that is induced by vertices and edges of $f_1, \ldots, f_i$.

- We denote $P_i$ (resp. $Q_i$) the right (resp. top) boundary of $G^i_{S-N}$ (resp. $G^i_{W-E}$).
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

S-N net $G_{S-N}$

$P_6$
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

S-N net $G_{S-N}$

$P_{13}$

16 - 4
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

Paths $P_i$ and $Q_j$ for any $i, j$ (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.
Rectangular Dual

- Paths $P_i$ and $Q_j$ for any $i, j$ (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.

Lemma 4

Let $v \in V$, $f$ and $g$ are the left and the right face of $v$. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex $v$ belongs to path $P_i$ if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof

- $f_{sn}(f) \leq i$ and $f_{sn}(g) \geq i + 1$
Rectangular Dual

- Paths $P_i$ and $Q_j$ for any $i, j$ (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.

**Lemma 4**

Let $v \in V$, $f$ and $g$ are the left and the right face of $v$. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex $v$ belongs to path $P_i$ if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

**Lemma 5**

Let $v \in V$, $f$ and $g$ are the faces below and above $v$ in $G_{W - E}$. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex $v$ belongs to path $Q_j$ if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:
Lemma 6
The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there **exists a vertex** over this box: \( u \in P_i \cap Q_j \)
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is at most one vertex over this box.
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is at most one vertex over this box.
Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is at most one vertex over this box

\[ x_1(u) \leq i \text{ and } i + 1 \leq x_2(u) \]
Rectangular Dual

**Lemma 6**

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

\[ x_1(u) \leq i \text{ and } i+1 \leq x_2(u) \]

\[ u \text{ belongs to } P_i \]
Rectangular Dual

### Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

**Proof:** Show that there is **at most one vertex** over this box

\[ x_1(u) \leq i \quad \text{and} \quad i+1 \leq x_2(u) \]

(Lemma 4)

\[ u \text{ belongs to } P_i \]

Similarly: \( v \in P_i, u \in Q_j, v \in Q_j \).
Rectangular Dual

**Lemma 6**

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

Paths $P_i$ and $Q_j$ intersect at two vertices $u$ and $v$.

\[
x_1(u) \leq i \text{ and } i+1 \leq x_2(u) \quad \text{(Lemma 4)}
\]

\[
u \text{ belongs to } P_i
\]

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$. 

18 - 9
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is at most one vertex over this box

\[ x_1(u) \leq i \text{ and } i+1 \leq x_2(u) \]

(Lemma 4)

\[ u \text{ belongs to } P_i \]

Similarly: \( v \in P_i, u \in Q_j, v \in Q_j \).

Paths \( P_i \) and \( Q_j \) intersect at two vertices \( u \) and \( v \).

Which is a contradiction to the property of paths \( P_i, Q_j \) except for the cases when:

(a) \( i = 0, j = 0 \), (b) \( i = \max f_{sn} - 1, j = 0 \), (c) \( i = 0, j = \max f_{we} - 1 \), (d) \( i = \max f_{sn} - 1, j = \max f_{we} - 1 \) (corner boxes).
Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$. 
Rectangular Dual

Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$

Proof

\[ u \rightarrow v \]
**Rectangular Dual**

**Lemma 7**

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$

**Proof**

![Diagram of a network with a function $f$ from $u$ to $v$.]
Rectangular Dual

Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$

Proof

- $x_2(u) = f_{sn}(f) = x_1(v)$

19 - 4
Lemma 7

Let \( G_{S-N} \) and \( G_{W-E} \). The following are true:

- If \( (u, v) \in G_{W-E} \) then \( x_2(u) = x_1(v) \);
- If there exist a directed path from \( u \) to \( v \) in \( G_{W-E} \) containing at least two edges, then \( x_2(u) < x_1(v) \)

Proof

\[ x_2(u) = f_{sn}(f) = x_1(v) \]
Rectangular Dual

Lemma 7
Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$

Proof

- $x_2(u) = f_{sn}(f) = x_1(v)$

19 - 6
Rectangular Dual

Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$

Proof

- $x_2(u) = f_{sn}(f) = x_1(v)$
- $x_2(u) = f_{sn}(f)$
- $x_1(v) = f_{sn}(g)$
- $x_2(u) = f_{sn}(f)$
- $x_1(v) = f_{sn}(g)$

19 - 7
Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$;
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{S-N}$ containing at least two edges, then $y_2(u) < y_1(v)$.
Rectangular Dual

Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$;
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{S-N}$ containing at least two edges, then $y_2(u) < y_1(v)$.

Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices $u$ and $v$ have a common segment if and only if there exists edge $(u, v)$ in the graph.

Proof:

19 - 9
Assume $R(u)$ and $R(v)$ have a common boundary.
Rectangular Dual

Assume $R(u)$ and $R(v)$ have a common boundary.

$x_1(v) \leq i, i + 1 \leq x_2(v)$ and $x_1(u) \leq i, i + 1 \leq x_2(u)$
Assume $R(u)$ and $R(v)$ have a common boundary.

$x_1(v) \leq i, i + 1 \leq x_2(v)$ and $x_1(u) \leq i, i + 1 \leq x_2(u)$

(Lemma 4)

$u, v$ belong to $P_i$
Assume $R(u)$ and $R(v)$ have a common boundary.

\[ x_1(v) \leq i, \quad i + 1 \leq x_2(v) \quad \text{and} \quad x_1(u) \leq i, \quad i + 1 \leq x_2(u) \]

(Lemma 4)

\[ u, v \text{ belong to } P_i \]

If path between $u$ and $v$ has at least 2 edges, then by Lemma 7,
\[ y_2(u) < y_1(v) \]
Rectangular Dual

Assume \( R(u) \) and \( R(v) \) have a common boundary.

\[ x_1(v) \leq i, i + 1 \leq x_2(v) \] \( \text{and} \) \[ x_1(u) \leq i, i + 1 \leq x_2(u) \]

(Lemma 4)

\( u, v \) belong to \( P_i \)

If path between \( u \) and \( v \) has at least 2 edges, then by Lemma 7,
\[ y_2(u) < y_1(v) \]

A contradiction to the hypothesis!
Rectangular Dual

- Assume there exists an edge \((u, v) \in G_{W-E}\).

- Let \(Q_j\) be the path of \(G_{W-E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j, j + 1 \leq y_2(u)\) and \(y_1(v) \leq j, j + 1 \leq y_2(v)\).

- By Lemma 7, \(x_2(u) = x_1(v)\).
Assume there exists an edge \((u, v) \in G_{W - E}\).

Let \(Q_j\) be the path of \(G_{W - E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j\), \(j + 1 \leq y_2(u)\) and \(y_1(v) \leq j\), \(j + 1 \leq y_2(v)\).

By Lemma 7, \(x_2(u) = x_1(v)\).
Rectangular Dual

- Assume there exists an edge \((u, v) \in G_{W-E}\).

- Let \(Q_j\) be the path of \(G_{W-E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j, j + 1 \leq y_2(u)\) and \(y_1(v) \leq j, j + 1 \leq y_2(v)\).

- By Lemma 7, \(x_2(u) = x_1(v)\)

Lemma 8 is proved!
Rectangular Dual

**Theorem**

Every PTP graph $G$ has a rectangular dual which can be computed in linear time.
Rectangular Dual

**Theorem**

Every PTP graph $G$ has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of $G$
- Compute a revised canonical ordering of $G$
- Traverse the graph and color the edges, construct $G_{S-N}$ and $G_{E-W}$
- Construct the duals $G^*_{S-N}$ and $G^*_{E-W}$ of $G_{S-N}$ and $G_{E-W}$, respectively
- Compute a topological ordering of $G^*_{S-N}$ and $G^*_{E-W}$
- Assign coordinates to the rectangles representing vertices.
Discussion

Proportional Cartogram. Source: http://www.ncgia.ucsb.edu
Discussion

- A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

- A rectangular layout is **area-universal** if and only if it is **one-sided**.

  [Eppstein et al. SIAM J. Comp. 2012]
A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

A rectangular layout is **area-universal** if and only if it is **one-sided**.

[Eppstein et al. SIAM J. Comp. 2012]

A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

A rectangular layout is **area-universal** if and only if it is **one-sided**.

De Berg et al. 2009: 40 sides

Kawaguchi et al. 2007: 34 sides

Biedl et al. 2011: 12 sides

Alam et al. 2011: 10 sides

Alam et al. 2013: 8 sides (matches the lower bound)
Discussion

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- A rectangular layout is **area-universal** if and only if it is **one-sided**.
  
  [Eppstein et al. SIAM J. Comp. 2012]

- Area universal **rectilinear** representation - possible for all planar graphs
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Discussion

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  [Eppstein et al. SIAM J. Comp. 2012]

```
one-sided
```

```
not one-sided
```

- Area universal **rectilinear** representation - possible for all planar graphs
  
  - De Berg et al. 2009: 40 sides
  - Kawaguchi et al. 2007: 34 sides
  - Biedl et al. 2011: 12 sides
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Discussion

A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

A rectangular layout is **area-universal** if and only if it is **one-sided**.

[Eppstein et al. SIAM J. Comp. 2012]

![One-sided and not one-sided layouts](image)

Area universal **rectlinear** representation - possible for all planar graphs

- De Berg et al. 2009: 40 sides
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Discussion

- Circular Arc Cartograms  [Kämper, Kobourov, Nöllenburg. IEEE PasViz 2013]

Source: http://cartogram.cs.arizona.edu