Exercise 1: Extended Canonical Ordering for 4-Connected Graphs

A planar graph \( G = (V, E) \) is called proper triangular planar (PTP, for short) if every interior face of \( G \) is a triangle and the exterior face of \( G \) is a quadrangle, and \( G \) has no separating triangles.

Let \( G = (V, E) \) be a PTP graph with vertices \( a, b, c, d \) on the outer face. A labeling \( v_1 = a, v_2 = c, v_3, \ldots, v_n = d \) of the vertices of \( G \) is called an extended canonical ordering of \( G \) if for every \( 4 \leq k \leq n \):

(i) The subgraph \( G_{k-1} \) induced by \( v_1, \ldots, v_{k-1} \) is biconnected and the boundary \( C_{k-1} \) of \( G_{k-1} \) contains the edge \((a, b)\), and

(ii) the vertex \( v_k \) is on the boundary of exterior face of \( G_{k-1} \), and its neighbors in \( G_{k-1} \) form a subinterval of the path \( C_{k-1} \setminus (a, b) \) with at least two elements. If \( k \leq n - 2 \), then \( v_k \) has at least two neighbors in \( G \setminus G_{k-1} \).

Let \( G = (V, E) \) be a PTP graph with vertices \( a, b, c, d \) on the outer face. Prove the following statements. We denote by \( G_C \) the graph that is induced by the vertices in the interior and on the boundary of a simple cycle \( C \).

(a) The graph obtained from \( G \) by the removal of the vertices \( c, d \) and all edges incident to them is biconnected.

(b) Let \( C = \{a = u_1, \ldots, u_k = b, a\} \) be a simple cycle of \( G \) such that \( c, d \notin C \). Let \( u_i \in C \), \( 2 \leq i \leq k - 1 \) such that no internal chord of \( C \) is incident to \( u_i \). Then the graph \( G_C \setminus \{u_i\} \) is biconnected.

(c) Let \( C \) be as above and let \( (v_i, v_j), 1 \leq i < j \leq k \), be an internal chord of \( C \). Then there exists a vertex \( v_l, i < l < j \) that is adjacent to at least two vertices of \( G \setminus G_C \).

Use the previous statements to prove the following lemma.

**Lemma 1** Every PTP graph \( G \) with four vertices \( a, b, c, d \) on the outer face has an extended canonical ordering such that \( v_1 = a, v_2 = b, v_{n-1} = c, v_n = d \).
Exercise 2: Construction of Rectangular Dual

Consider the graph $G$ of the figure below. Check whether $G$ satisfies the necessary conditions to have a rectangular dual. In affirmative, construct a rectangular dual of $G$.

Exercise 3: Contact Representation of Maximal Planar Graphs

The figure below gives an example of contact representation of a planar graph with T-shapes. Prove the following Lemma.

Lemma 2 Every maximal planar graph admits a contact representation with T-shapes.

Hint: Use canonical ordering in the way similar to the construction of a visibility representation (Exercise Sheet 3).