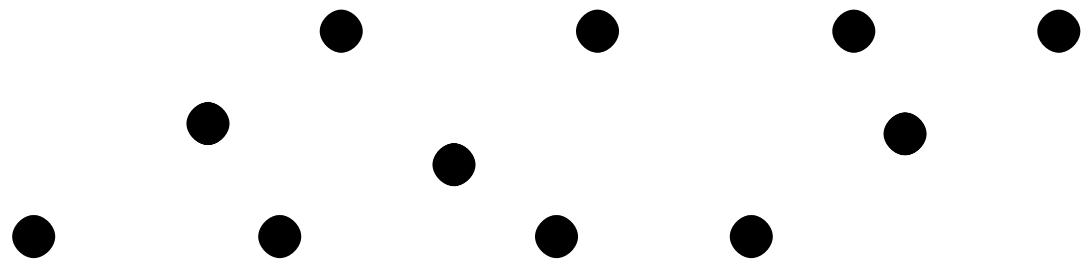
# Computational Geometry – Exercise
## Delaunay Triangulation

LEHRSTUHL FÜR ALGORITHMIK I · INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Benjamin Niedermann
09.12.2015

# Triangulation of a Point Set

**Def.:** A **triangulation** of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with a vertex set $P$.



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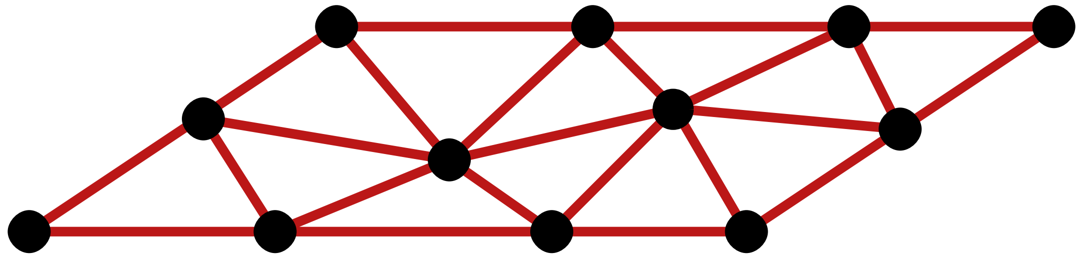

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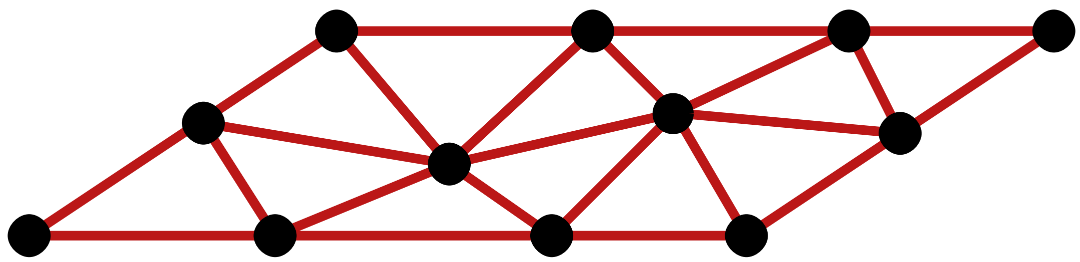

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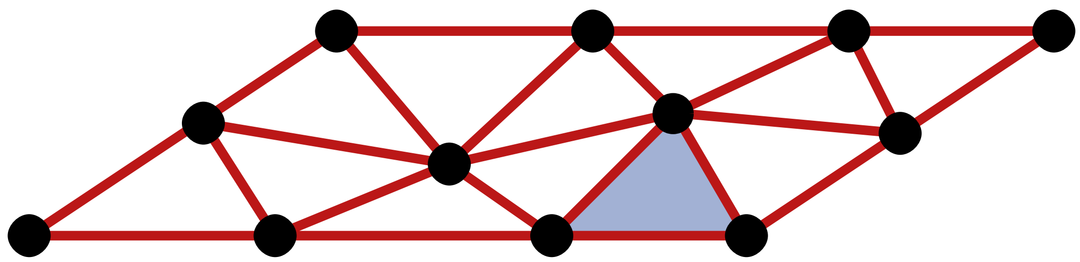

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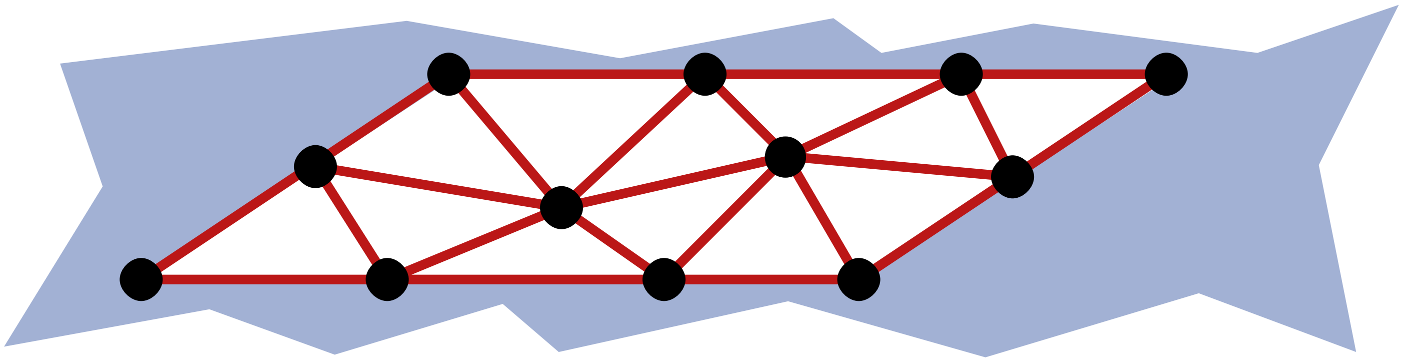

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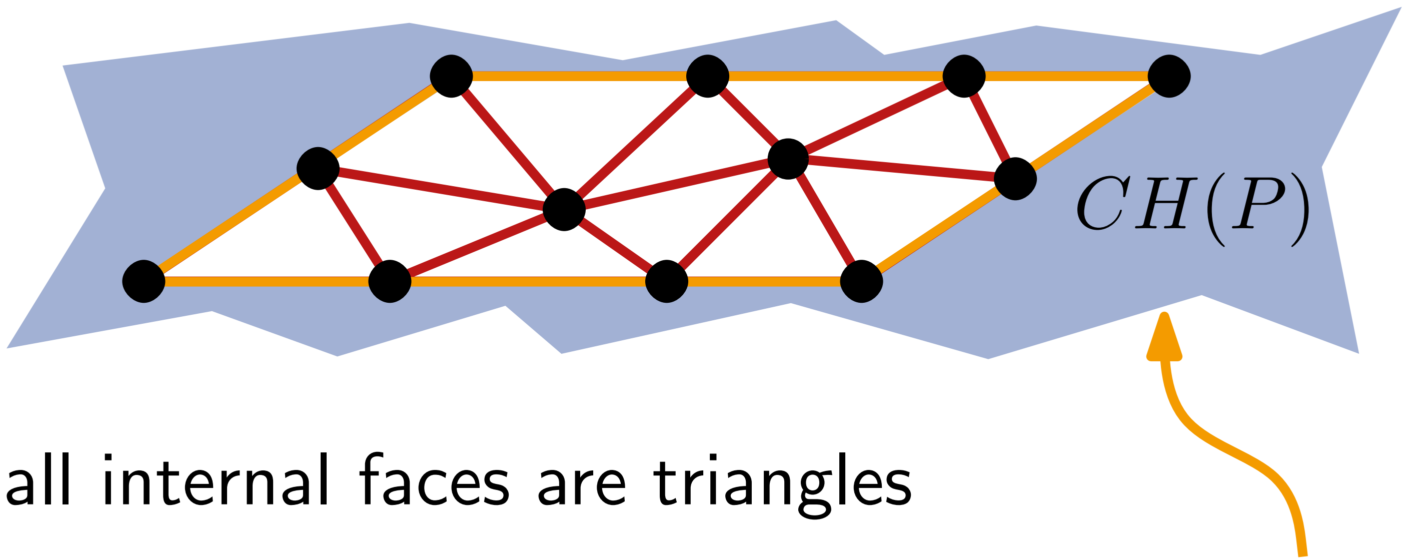


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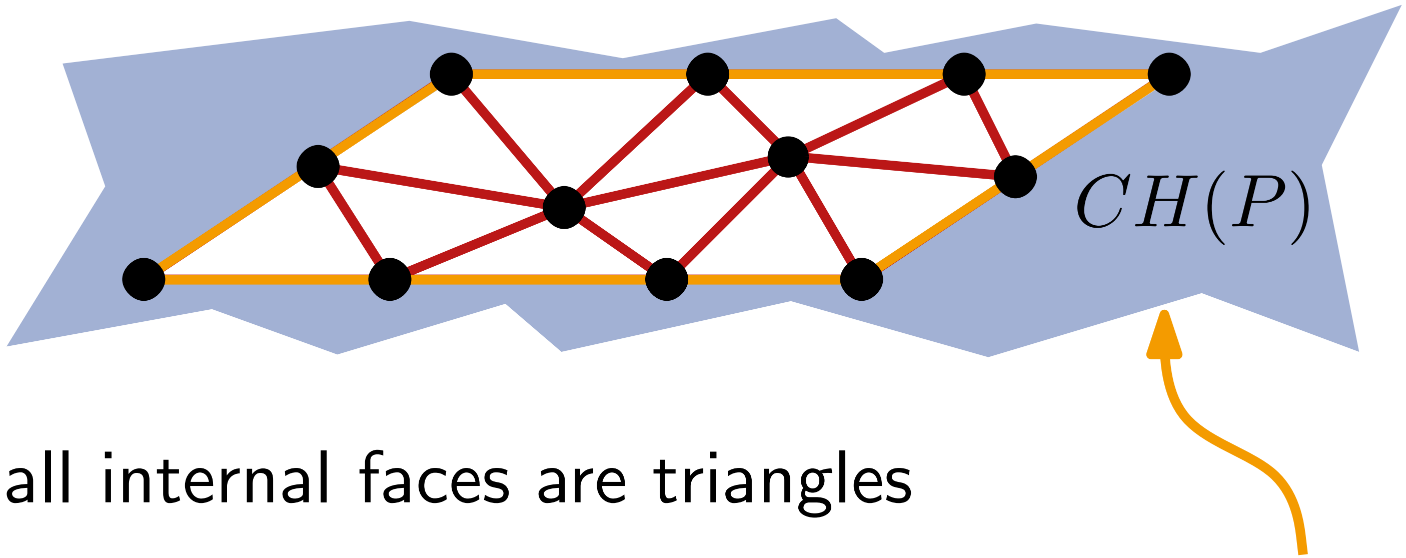


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**Theorem 1:** Let $P$ be a set of $n$ points, not all collinear. Let $h$ be the number of points in $CH(P)$.

Then any triangulation of $P$ has $(2n - 2 - h)$ triangles and $(3n - 3 - h)$ edges.

# Exercise 5

**Problem:**

Let $P \subset \mathbb{R}^2$ be a set of $n$ points.

a) There are at most $2^{\binom{n}{2}}$ triangulations of $P$.

b) Example: $P$ such that for each triangulation there is at least one point having degree $n-1$.

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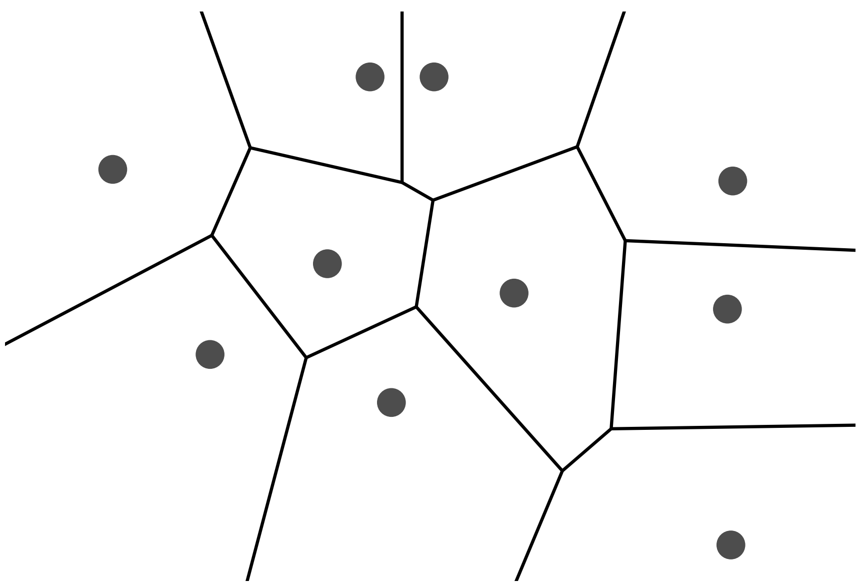
# Delaunay-Triangulation

Let $\mathrm{Vor}(P)$ be the Voronoi-Diagram of a point set $P$.

**Def.:** The graph $\mathcal{G} = (P, E)$ with
$E = \{pq \mid \mathcal{V}(p)$ and $\mathcal{V}(q)$ are adjacent$\}$
is called **dual graph** of $\mathrm{Vor}(P)$.



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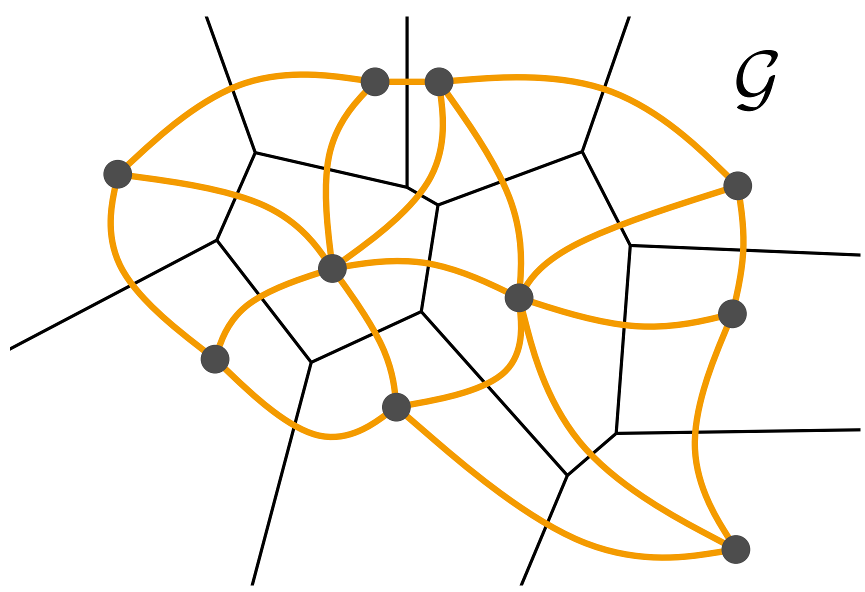

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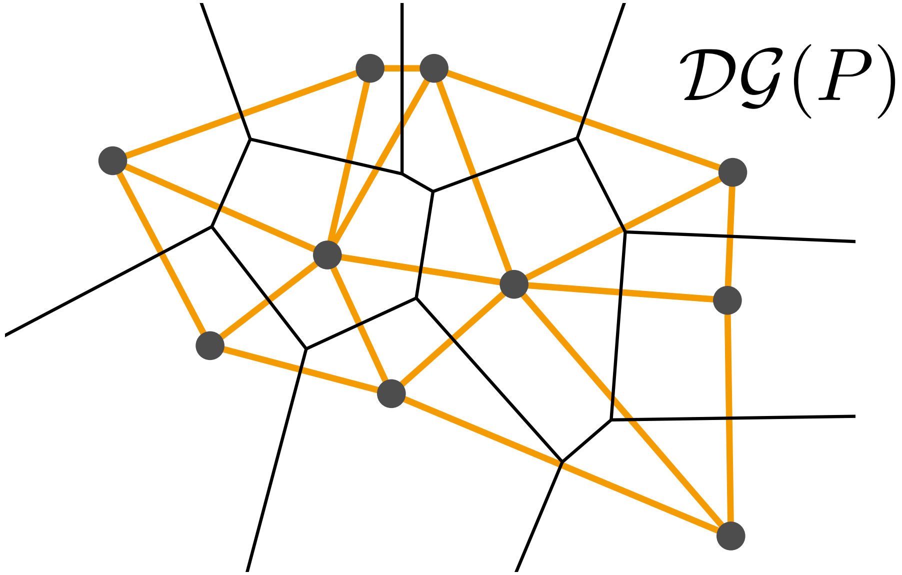
**Def.:** The straight-line drawing of $\mathcal{G}$ is called
**Delaunay-Graph** $\mathcal{DG}(P)$.



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Theorem about Voronoi-Diagram:

- point $q$ is a Voronoy-vertex
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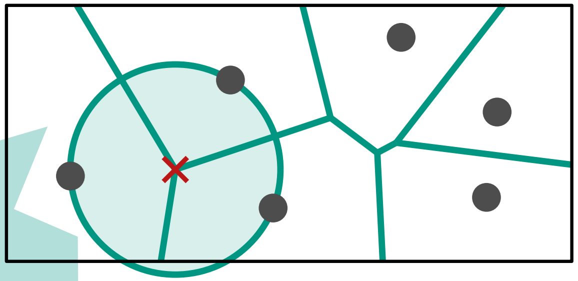

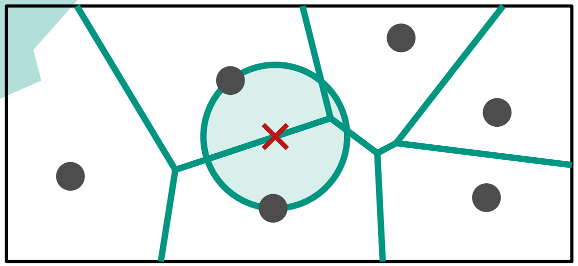

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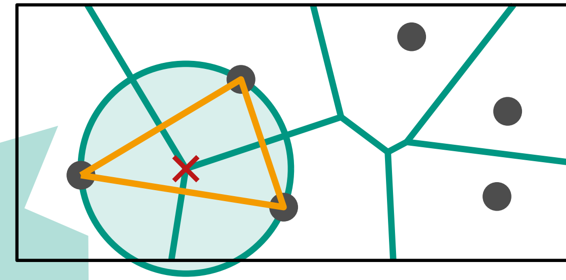

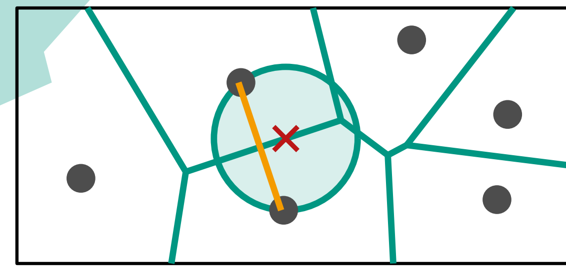

**Theorem 4:** Let $P$ be a set of points.

- Points $p, q, r$ are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow$
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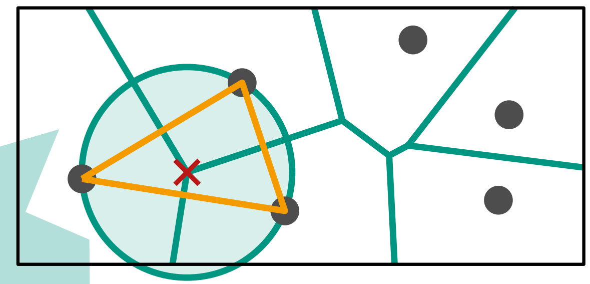

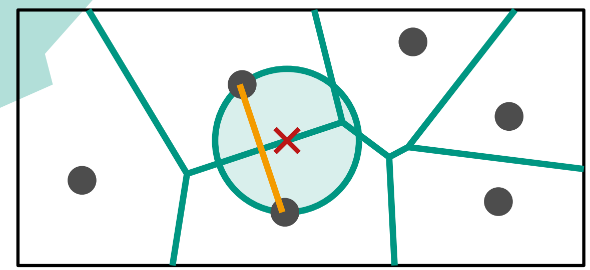

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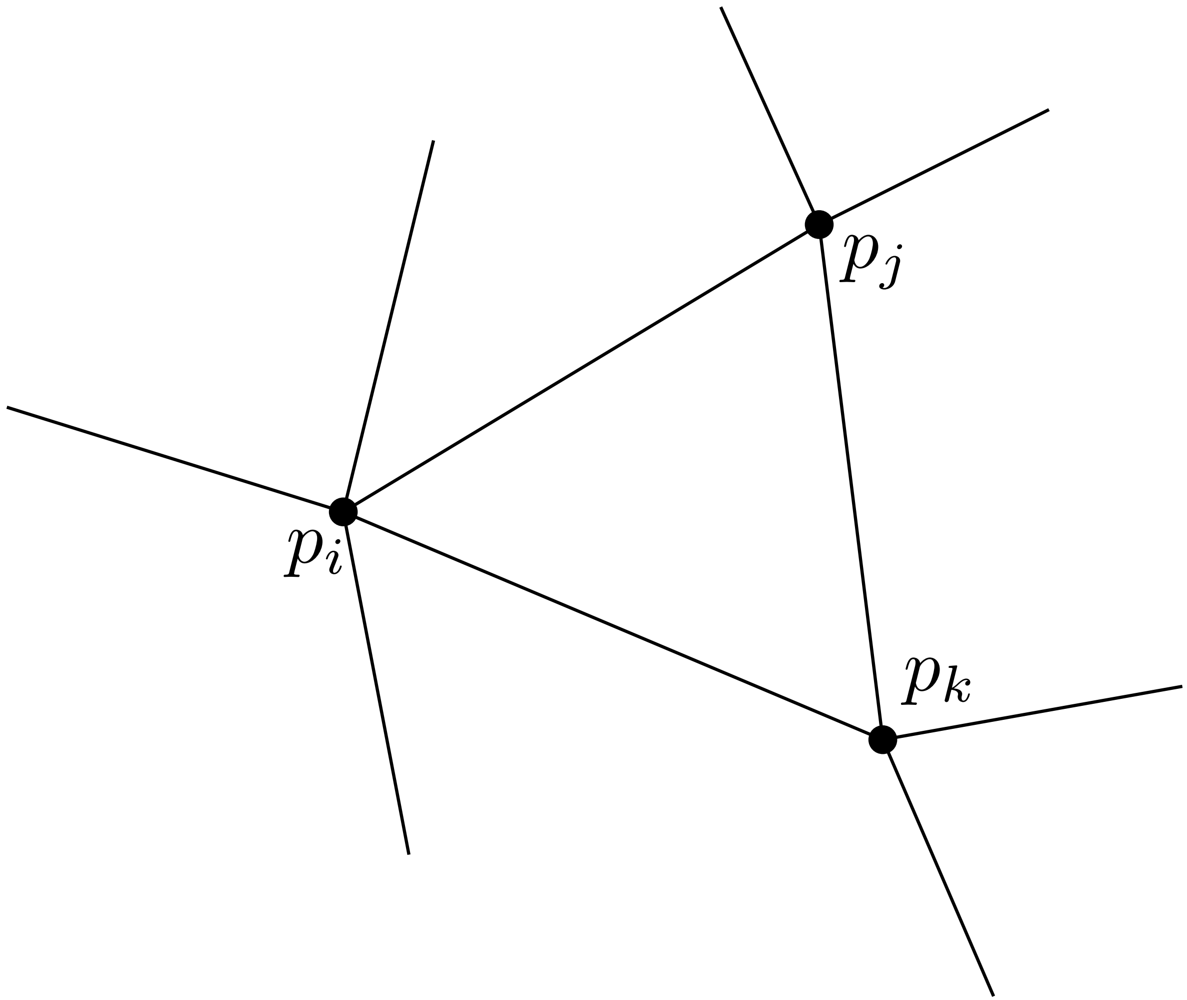

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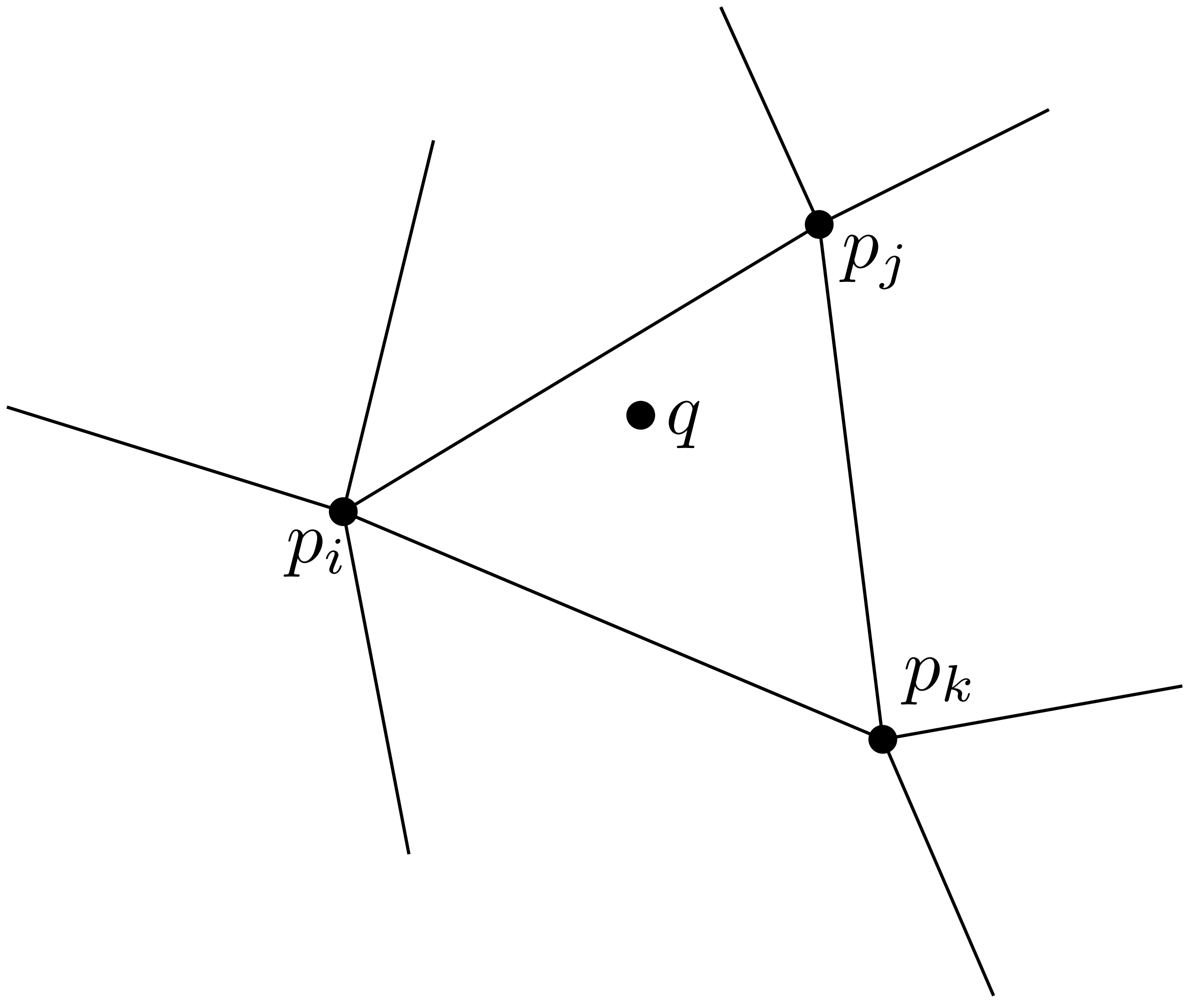

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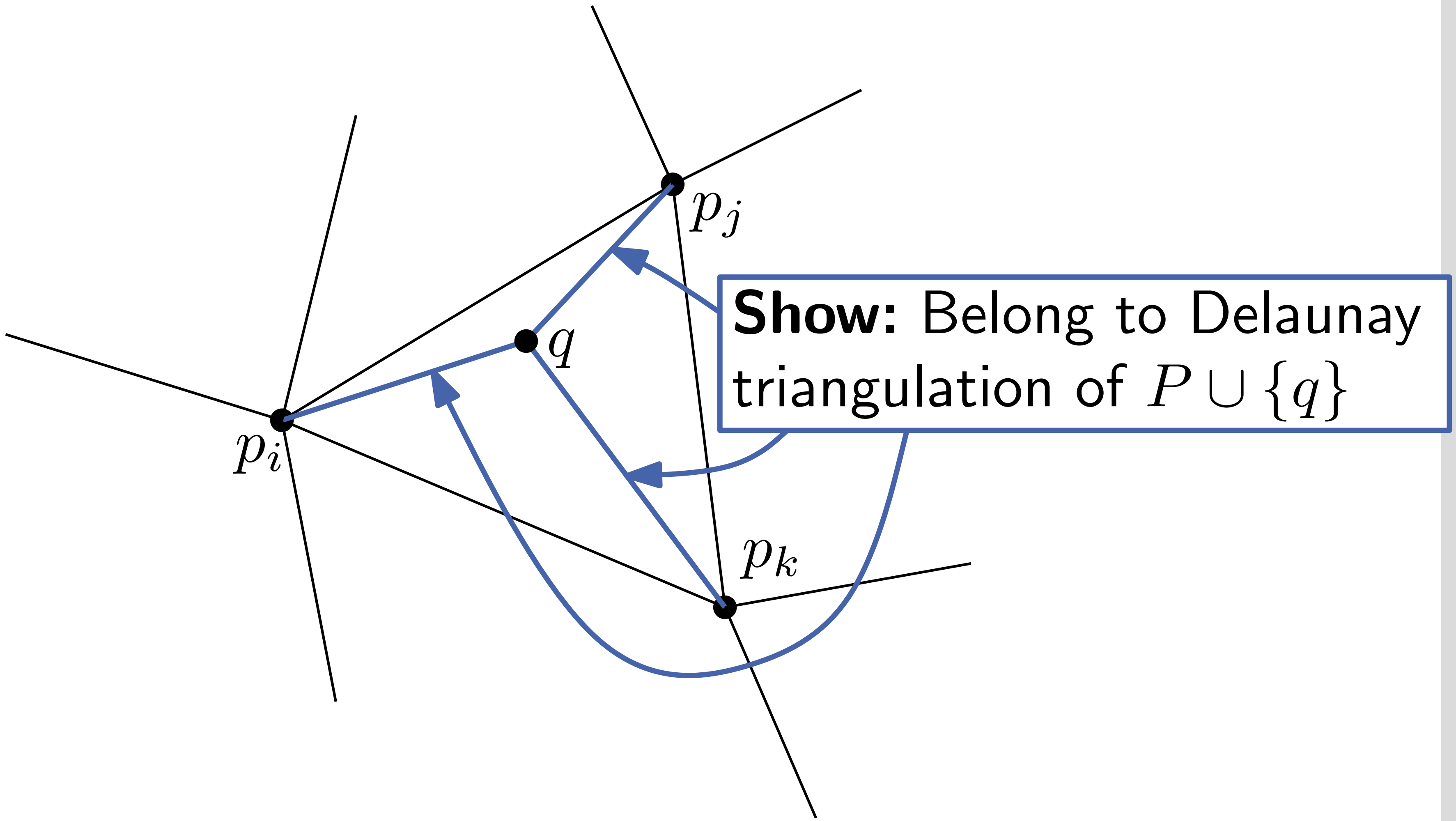

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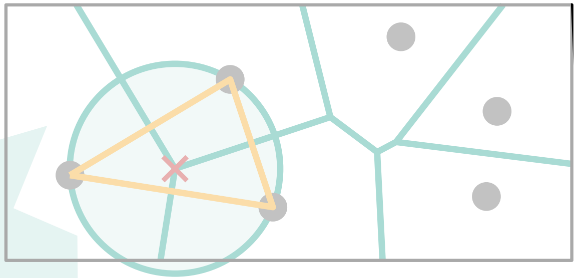

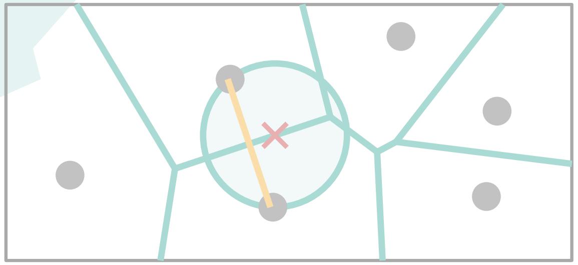

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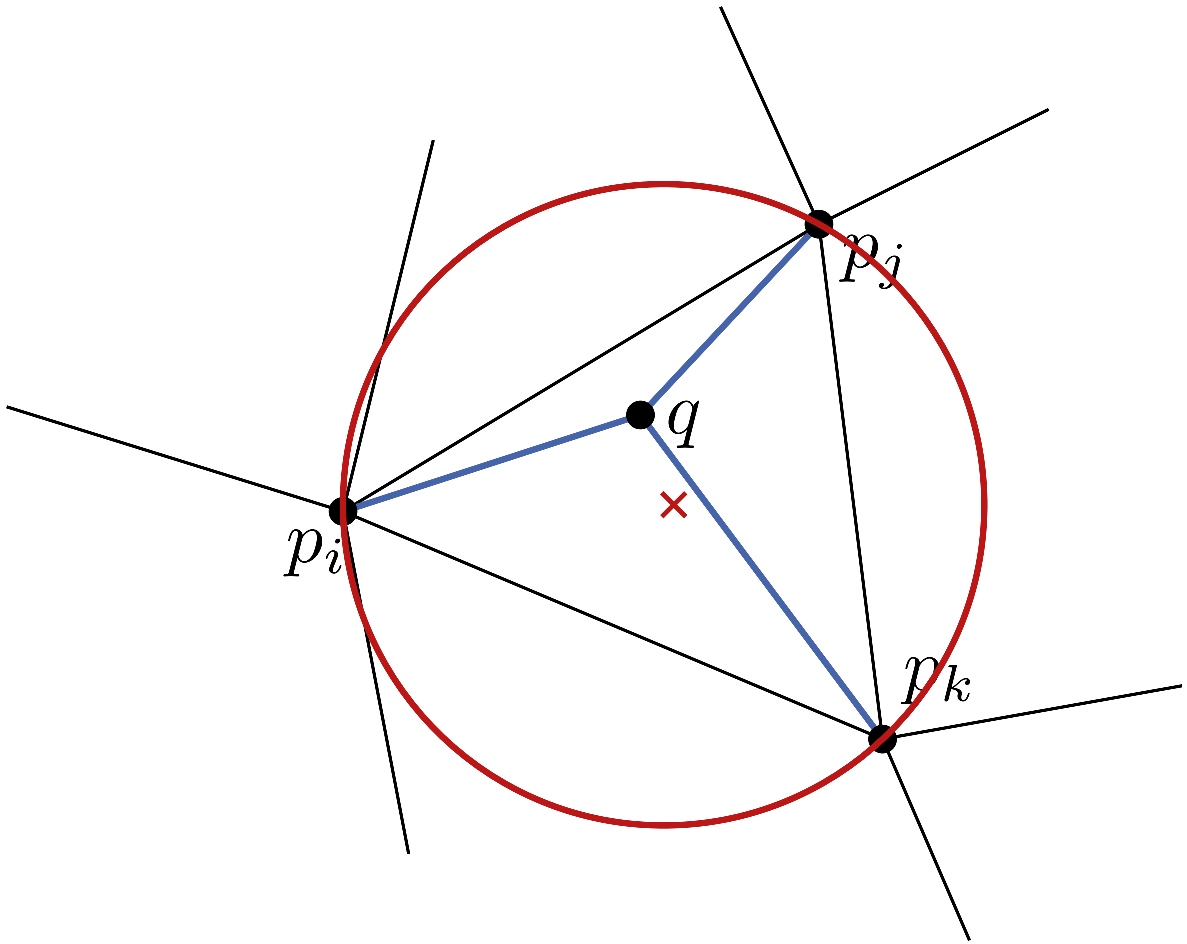


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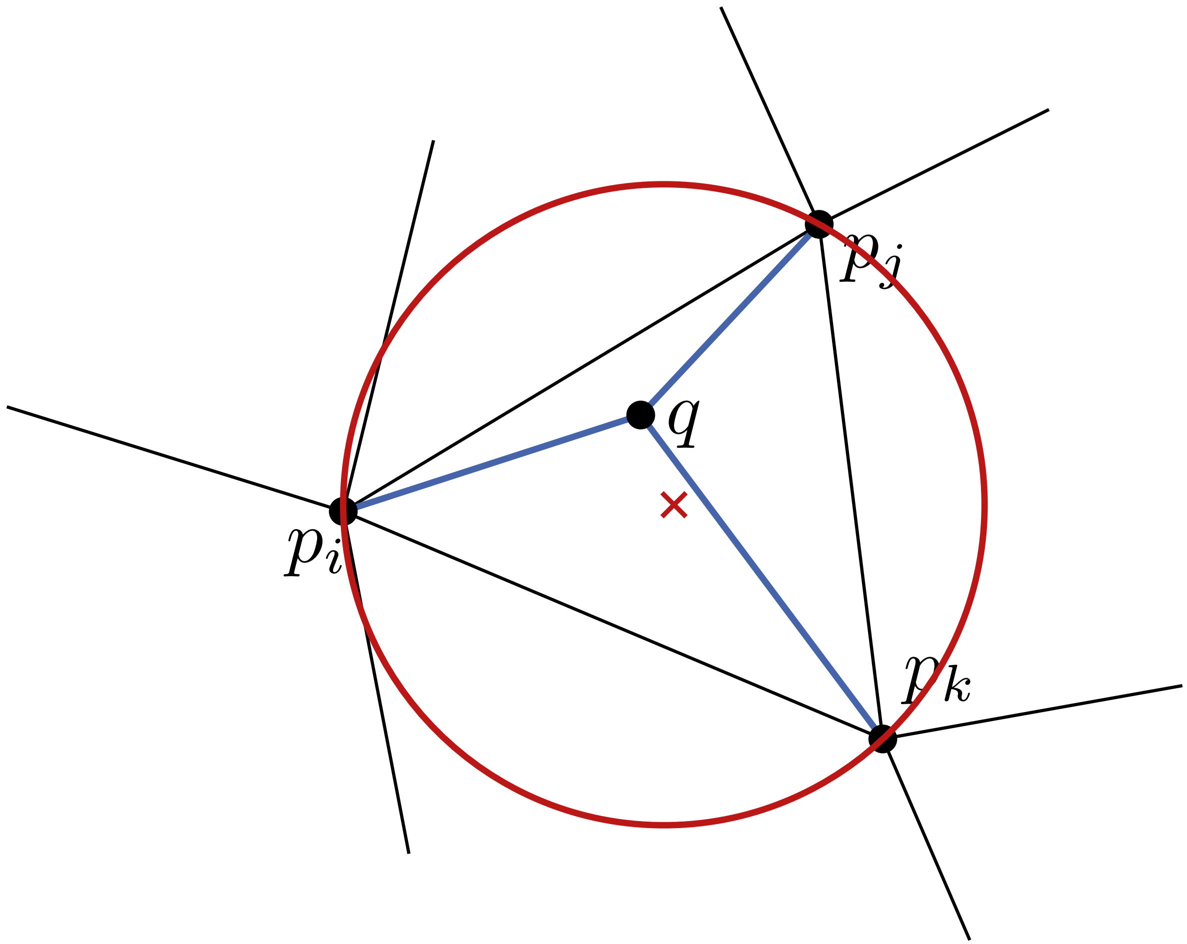


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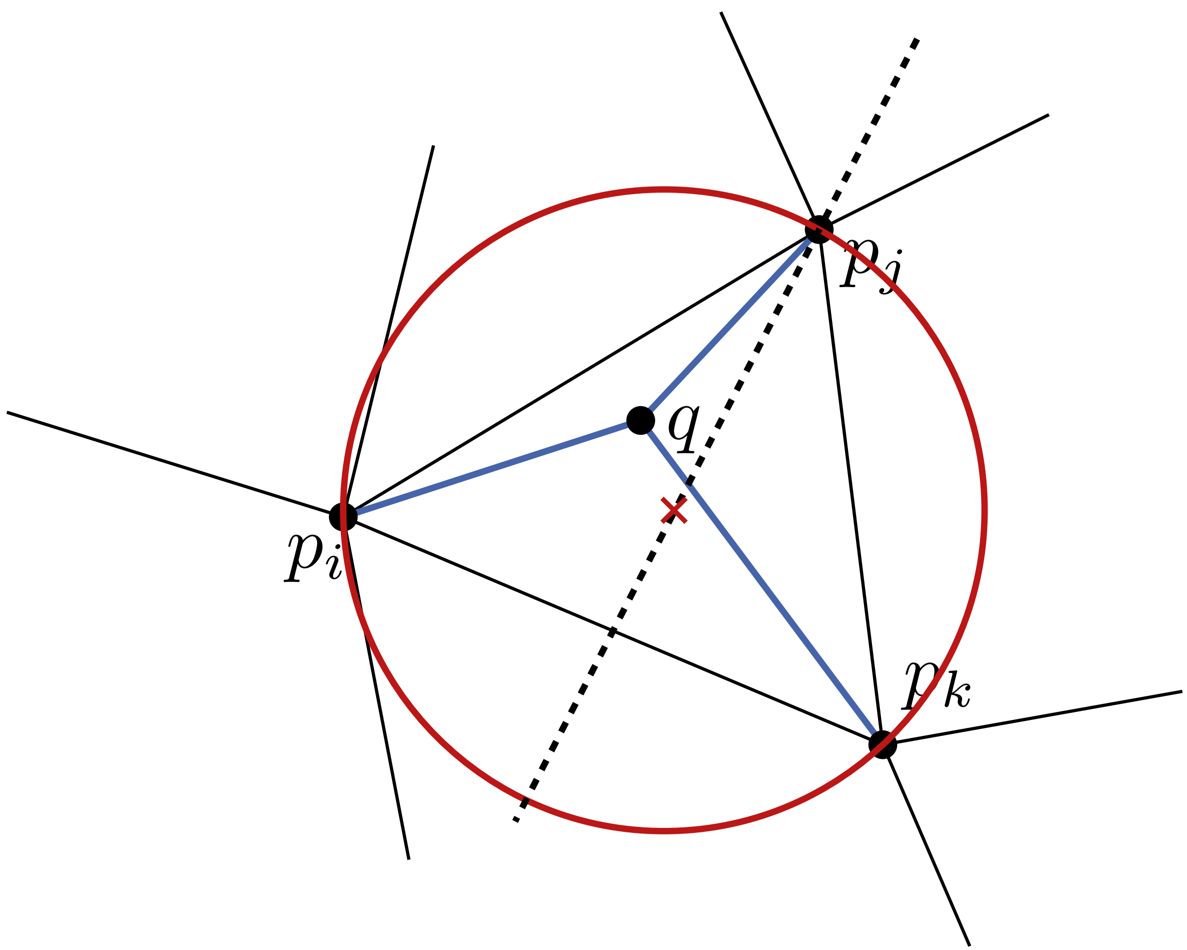


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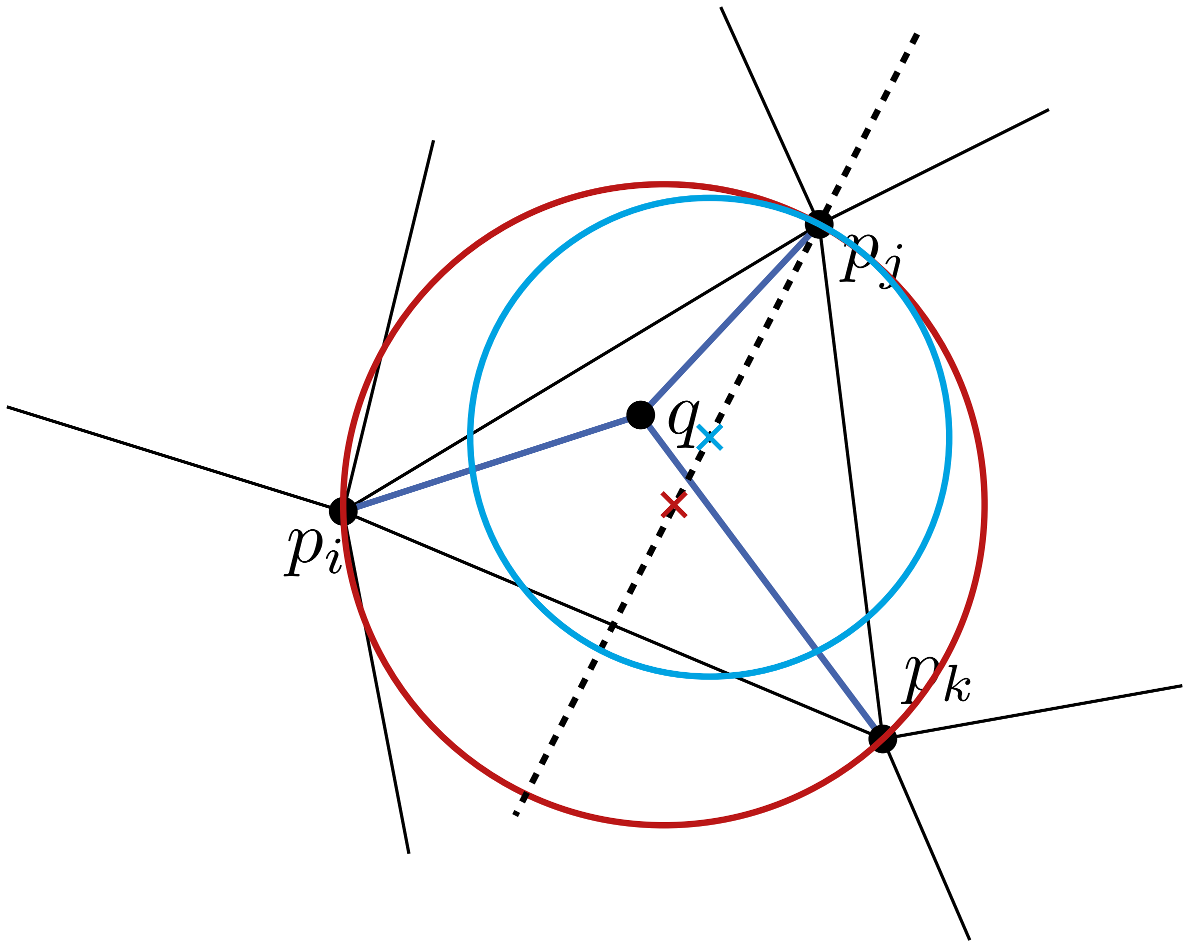


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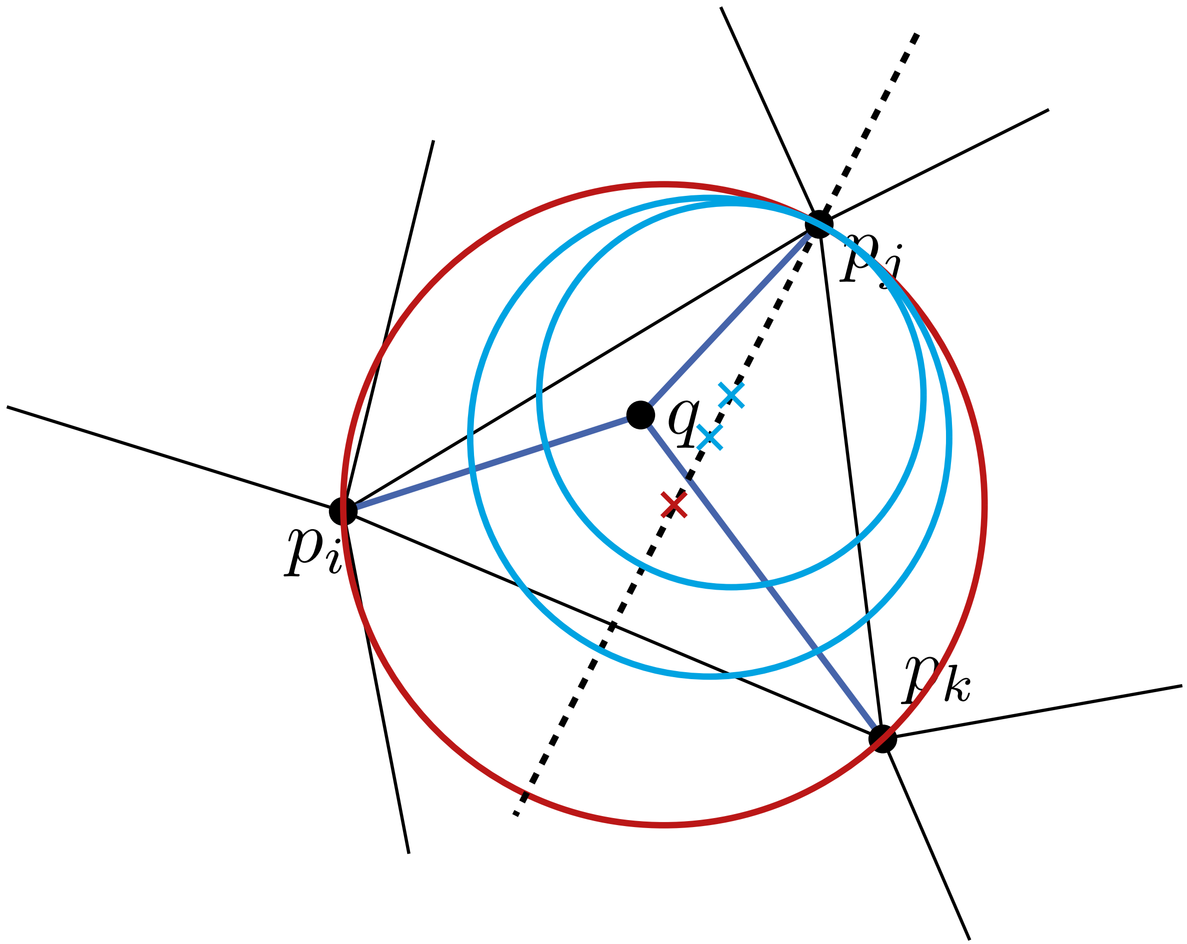


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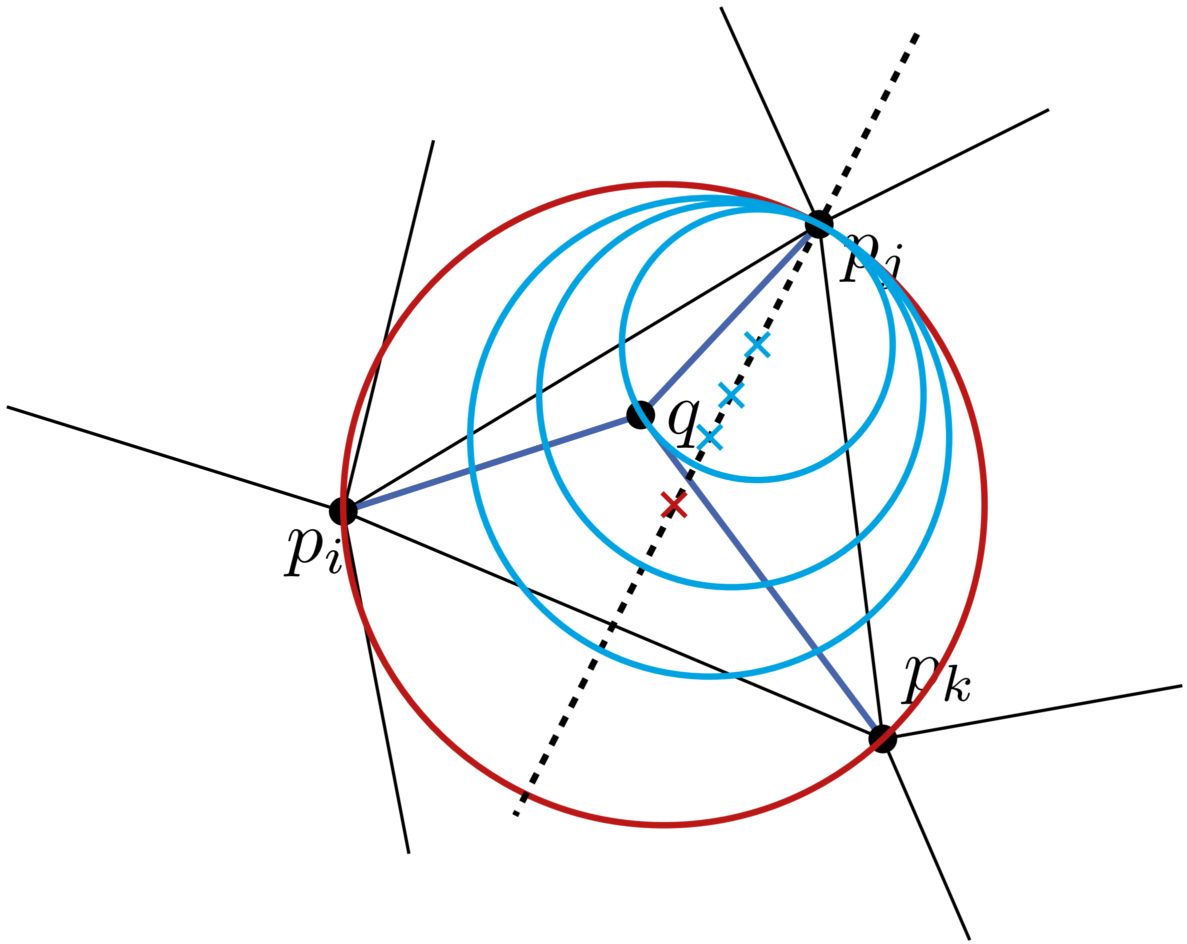


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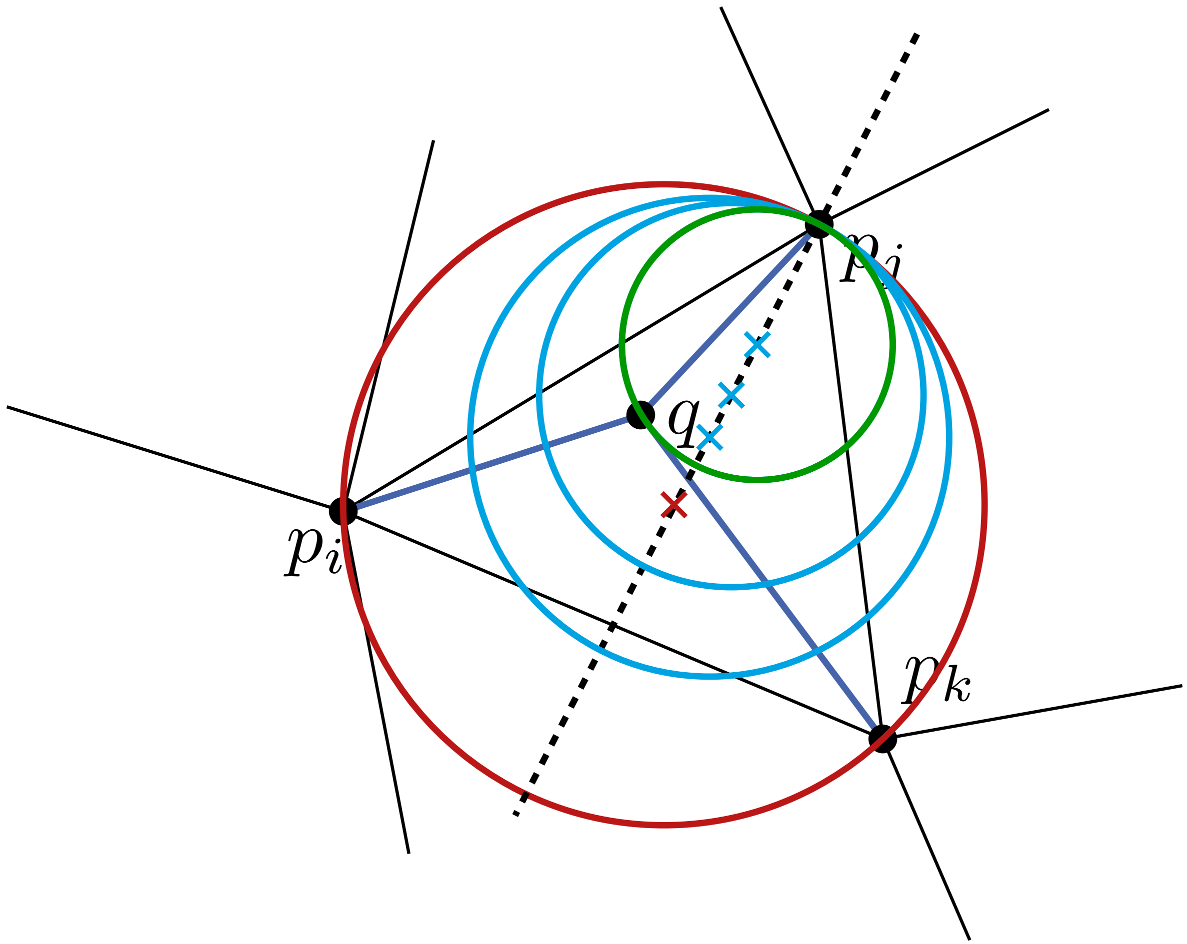


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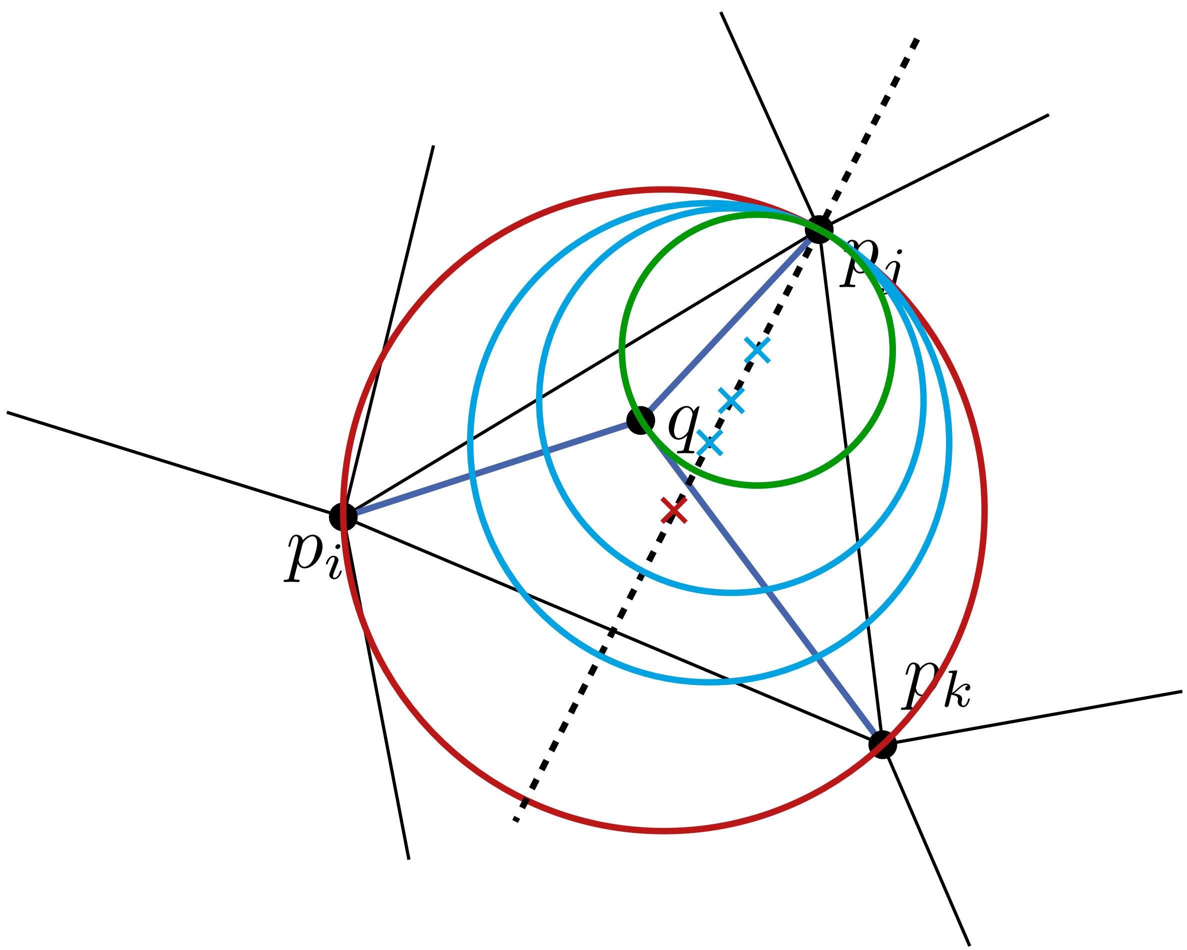


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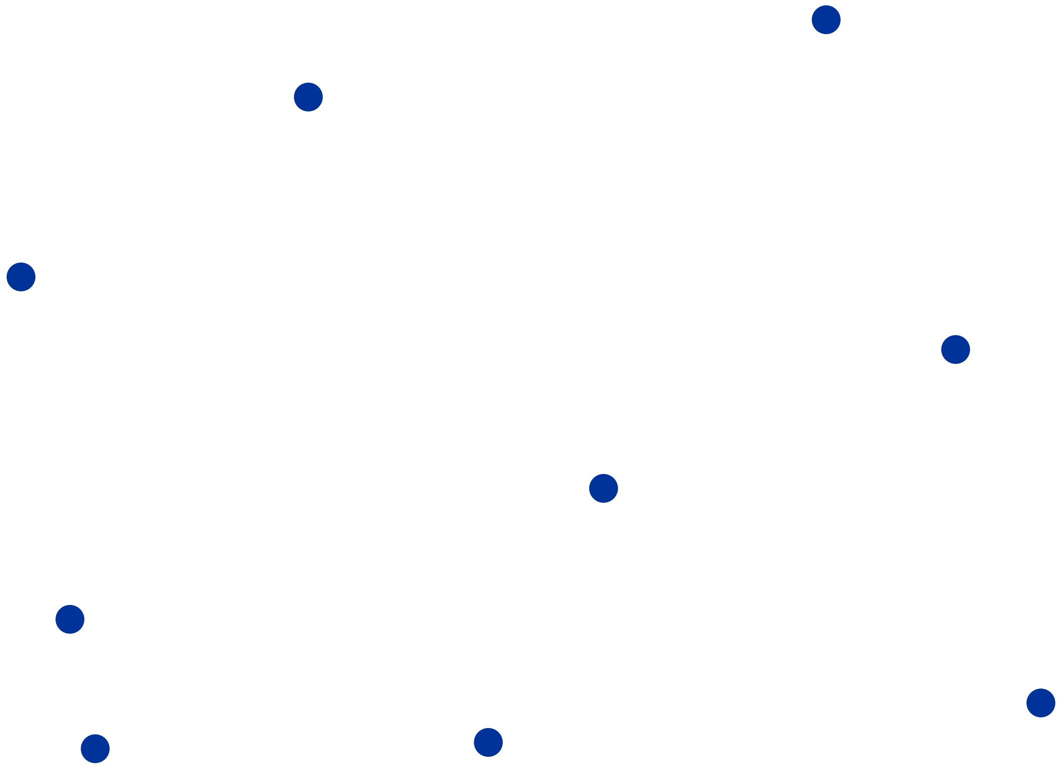

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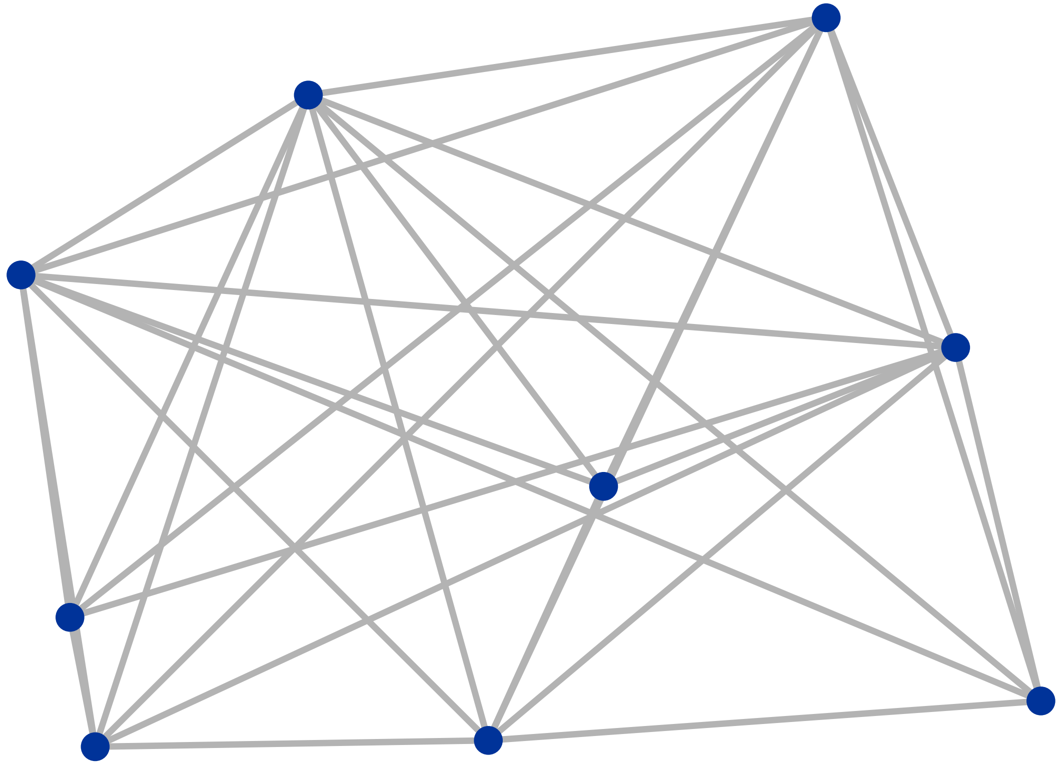

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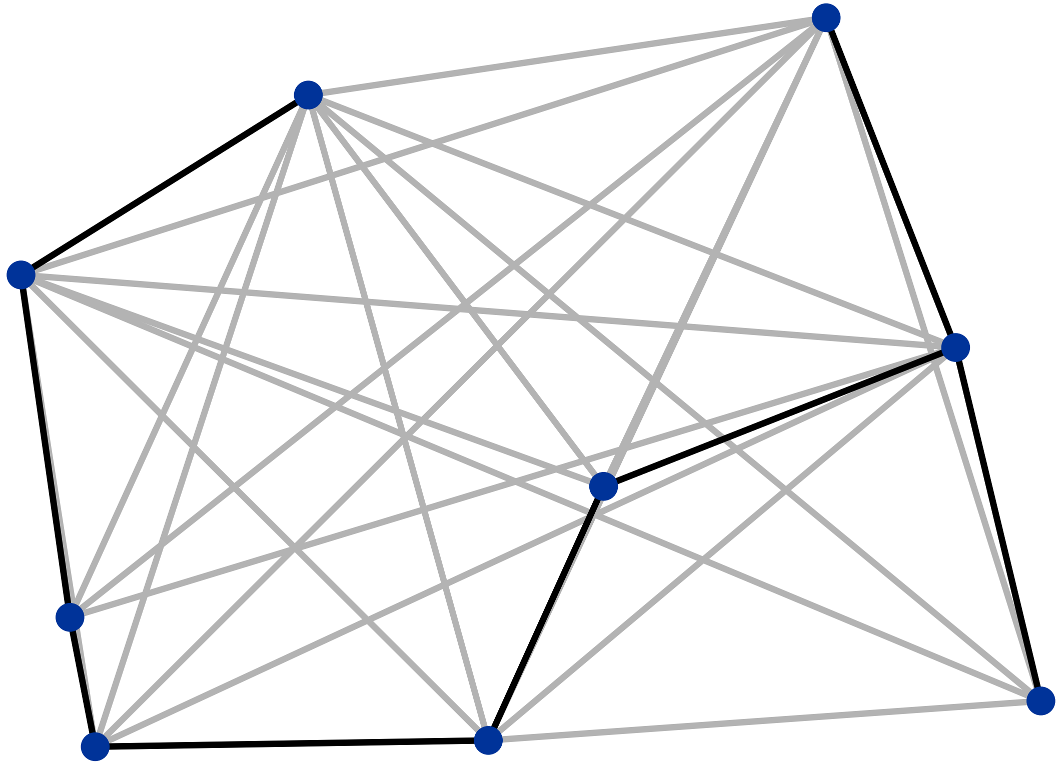

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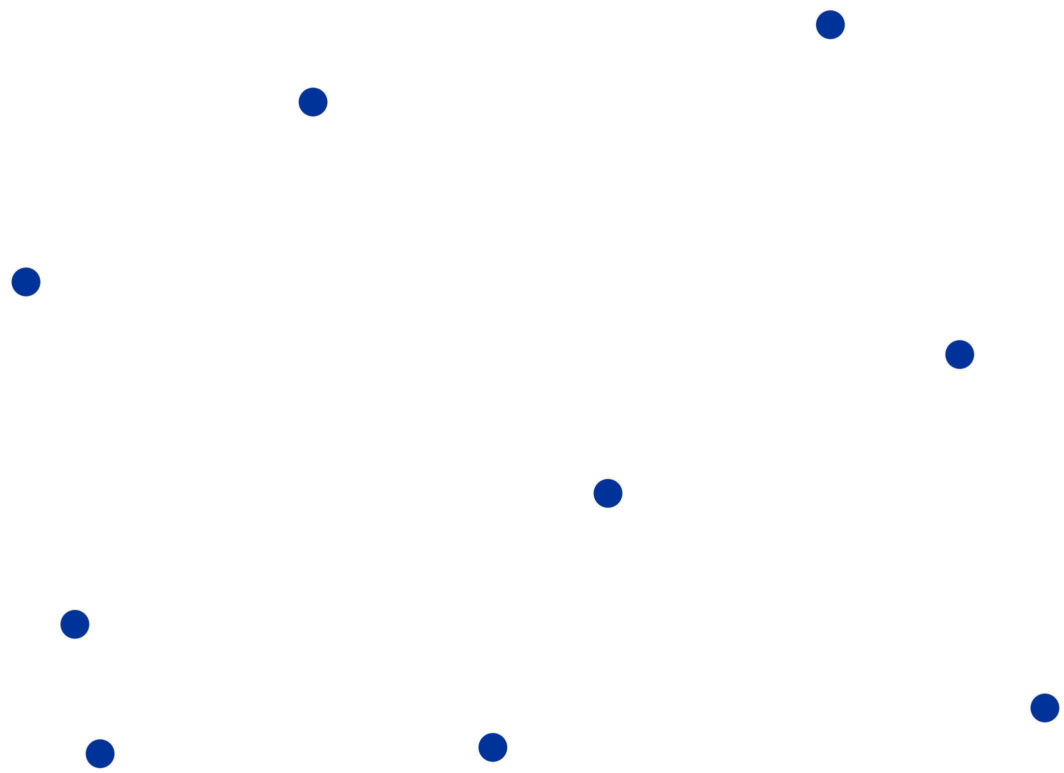

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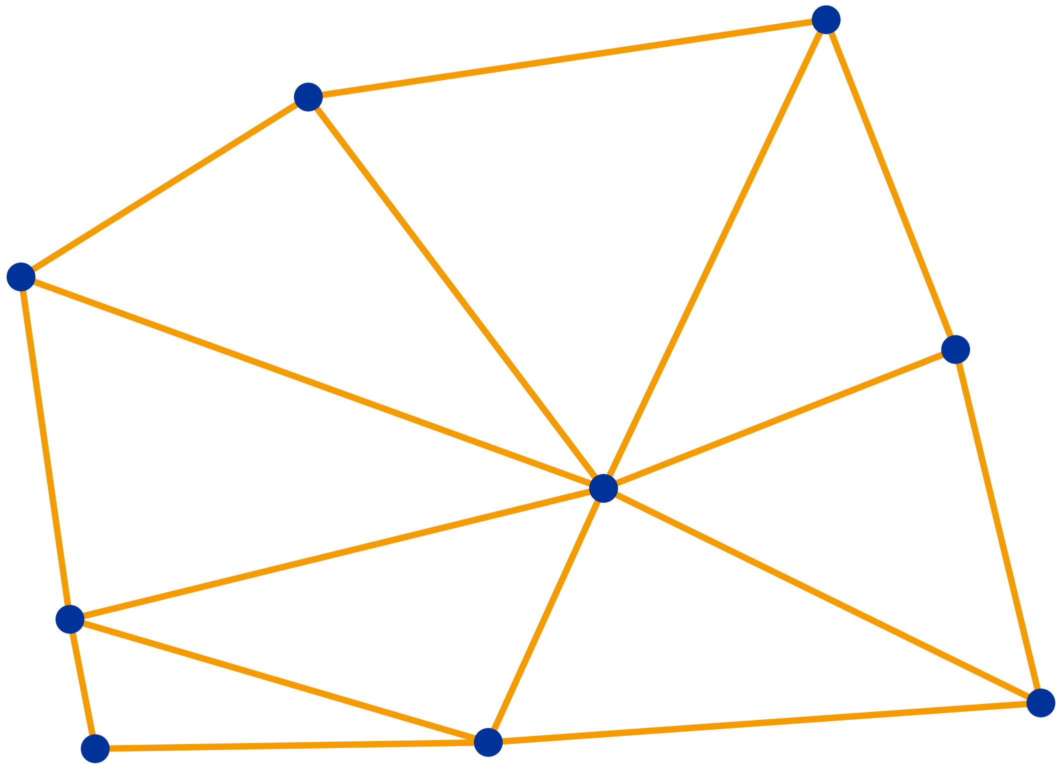

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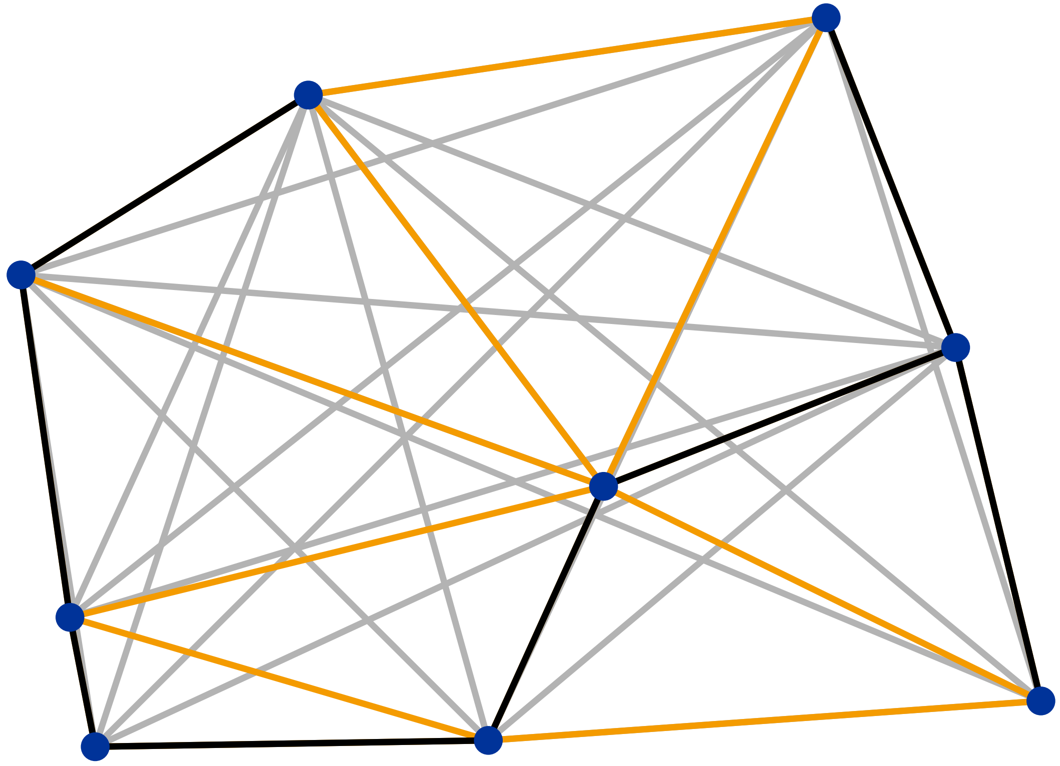

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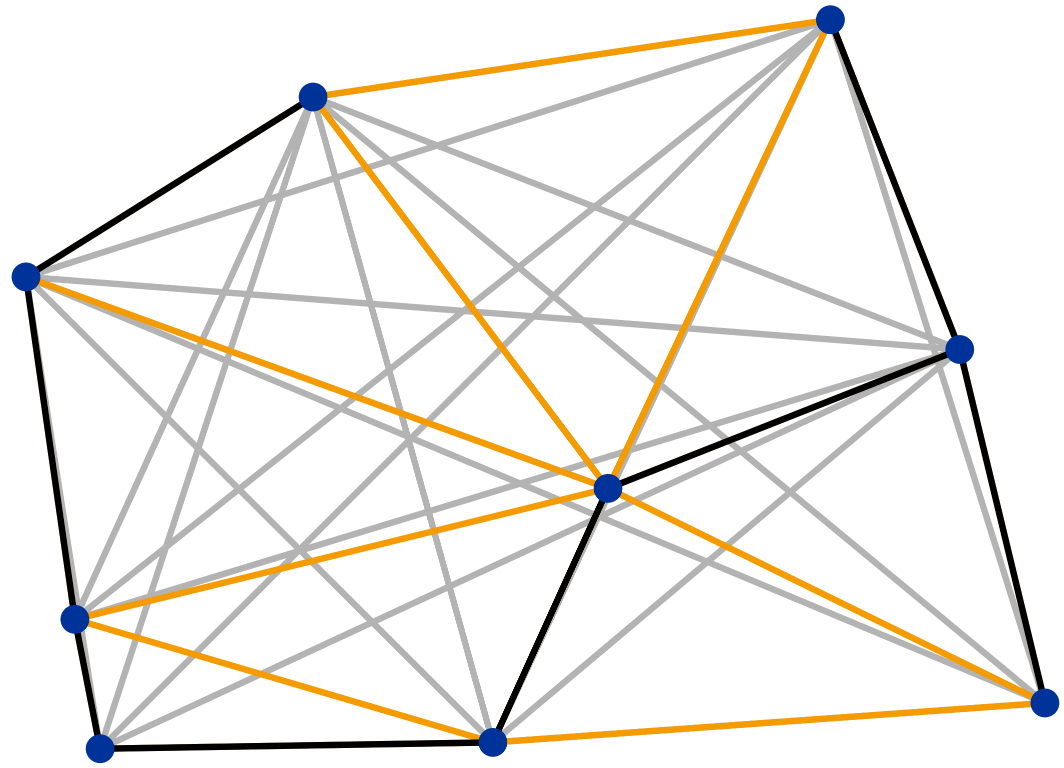

Show that edges of EMST form subset of the edges of the Delaunay-Graph

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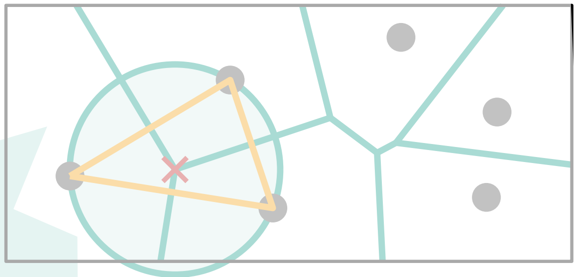

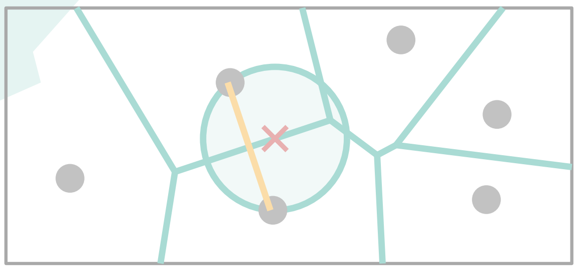

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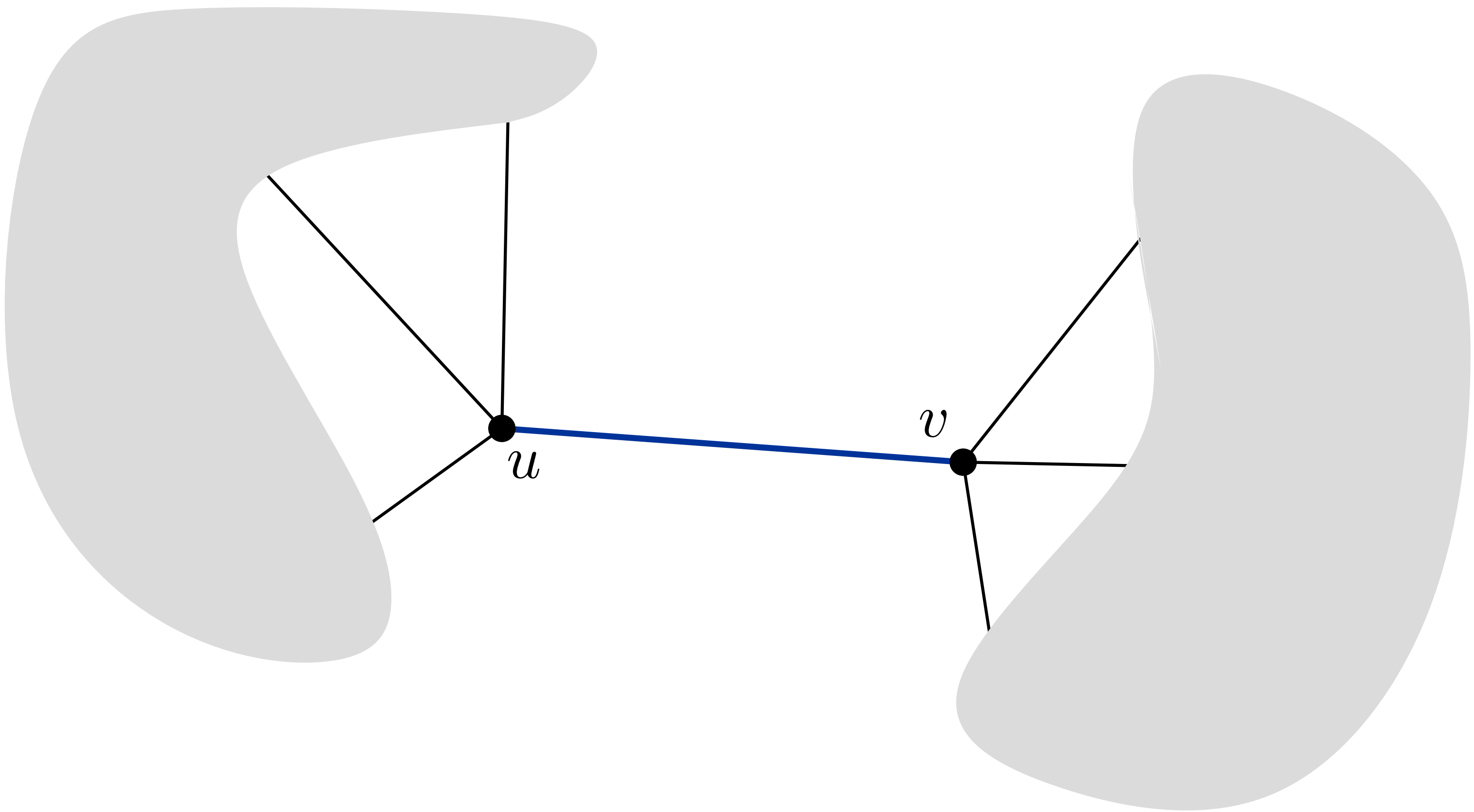


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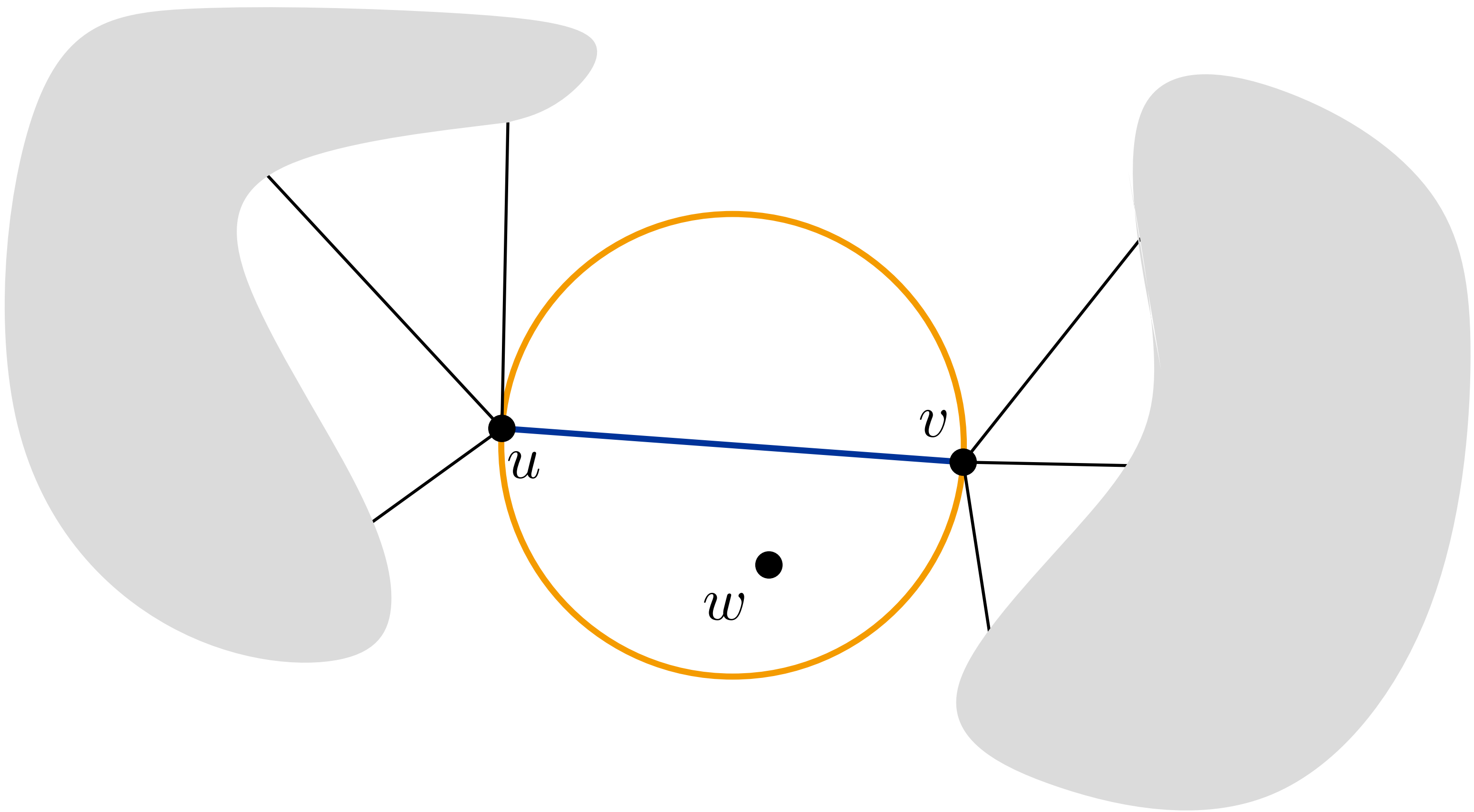


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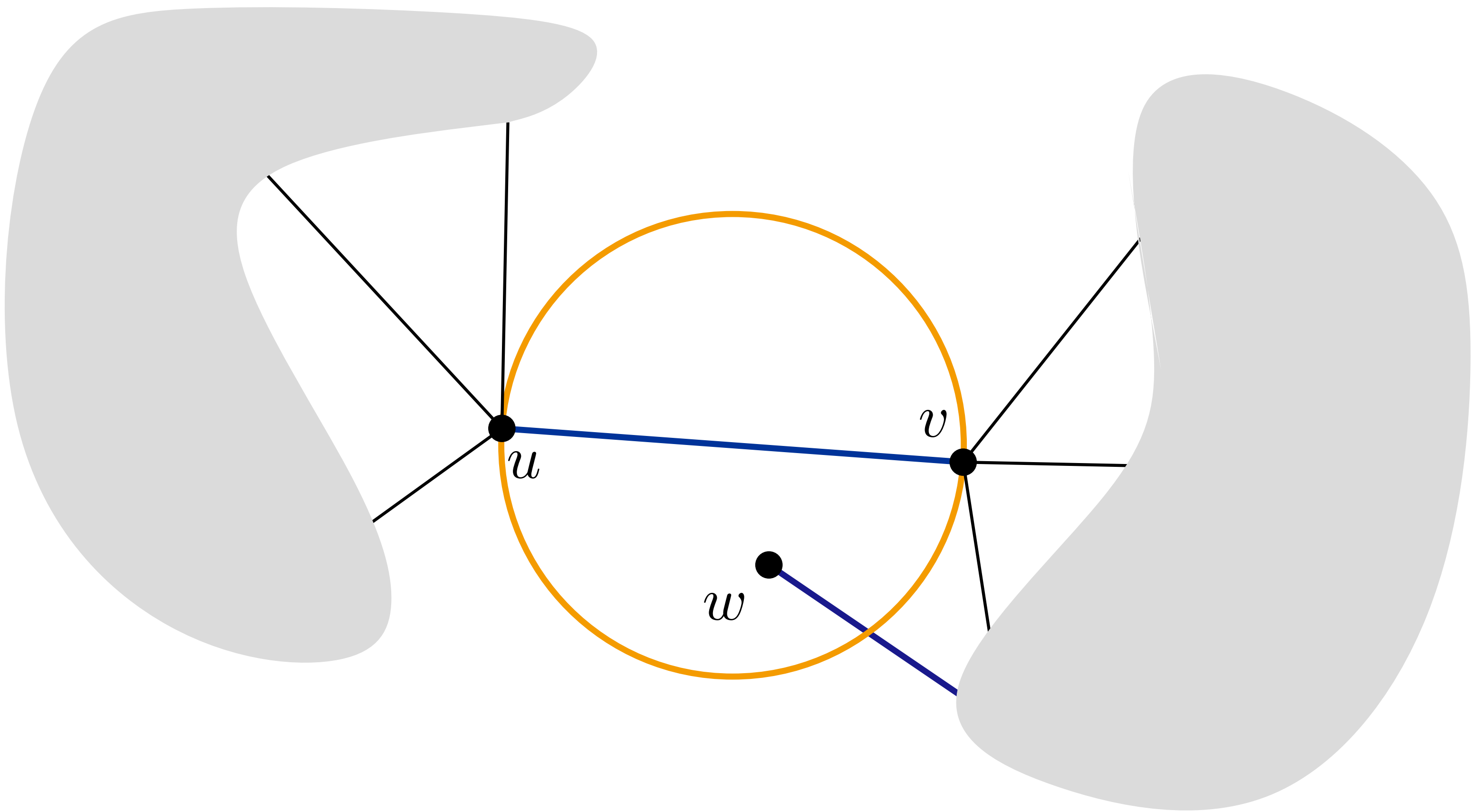


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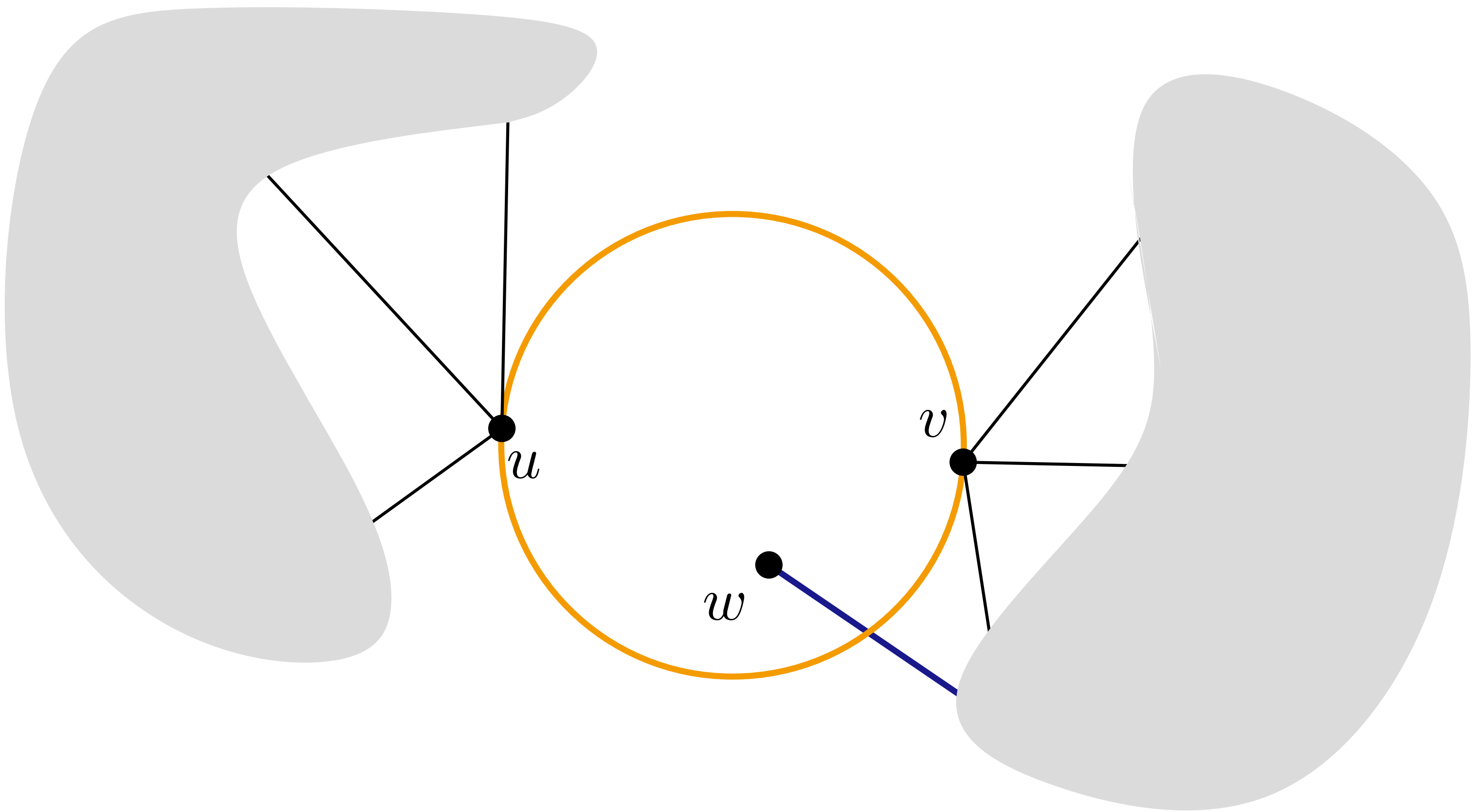


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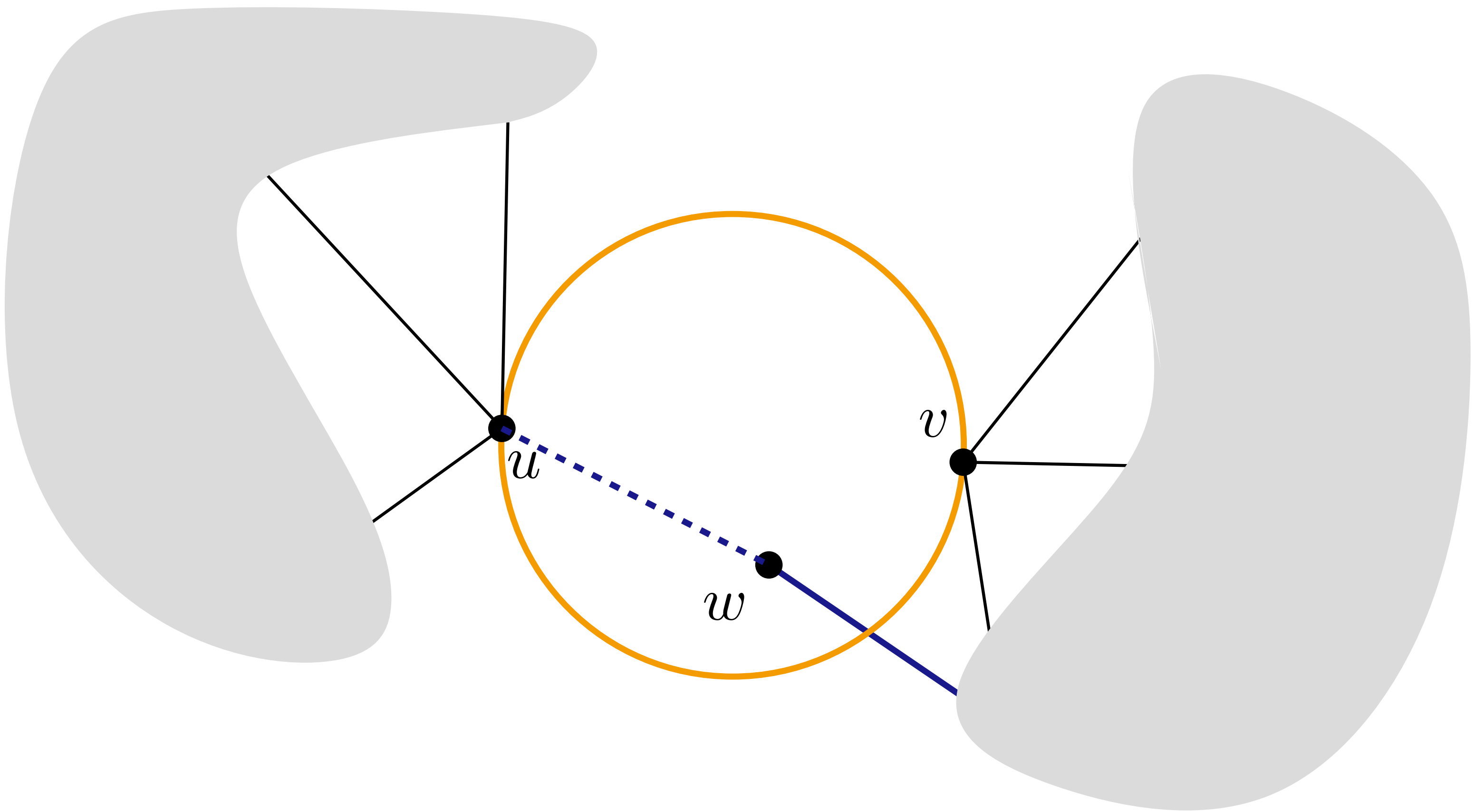


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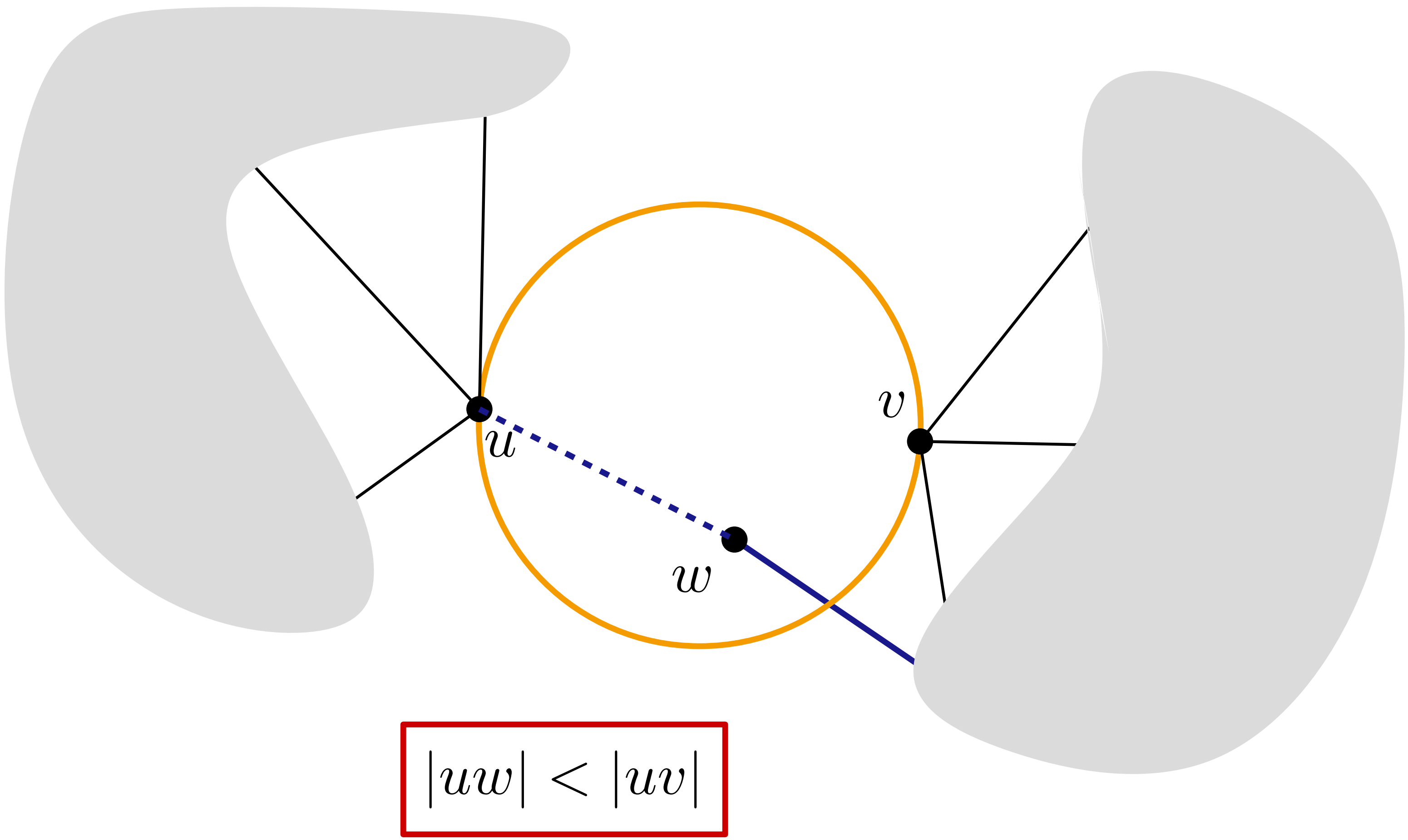


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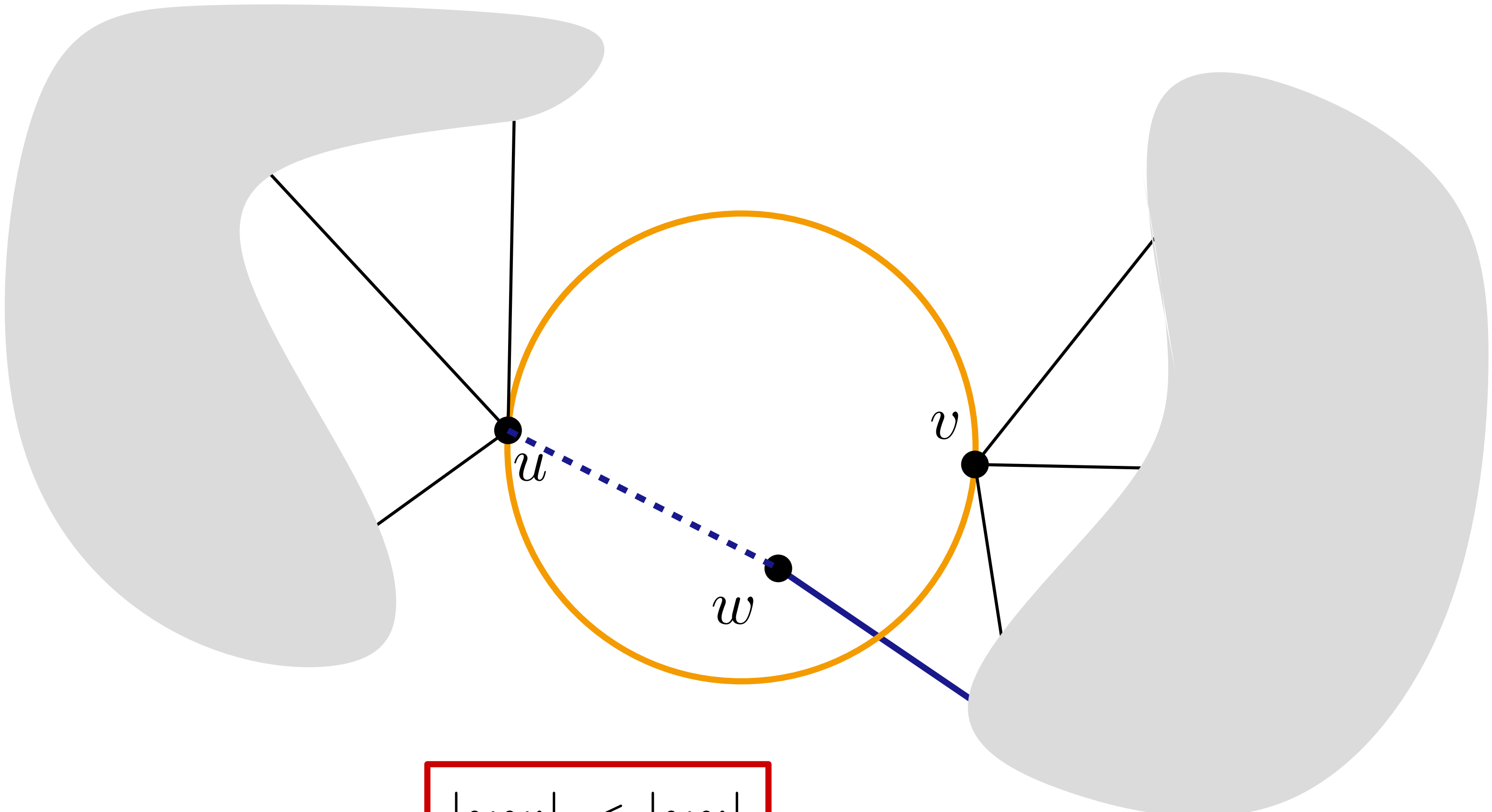


$$|uw| < |uv|$$

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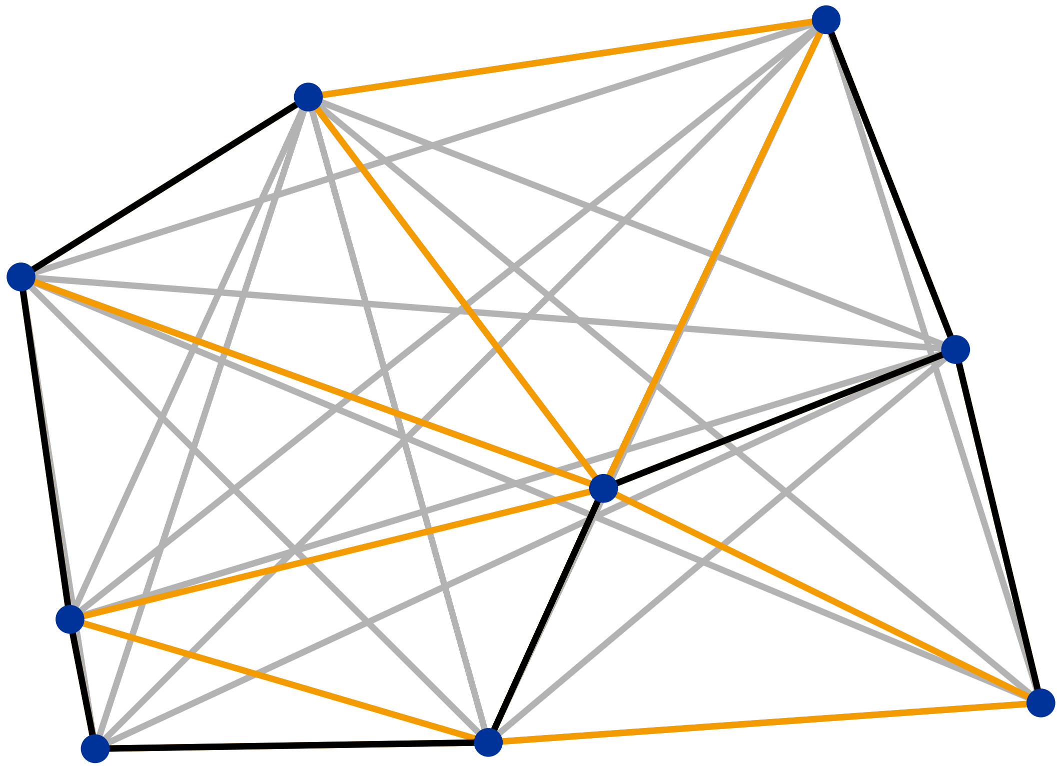
# Exercise 3

## Euclidean Minmium Spanning Tree



Computation of EMST in $\mathcal{O}(n \log n)$?

# Exercise 4

- Gabriel Graph: $p, q$ connected by edge, if circle $C_{p,q}$ with diameter $|pq|$ is empty.

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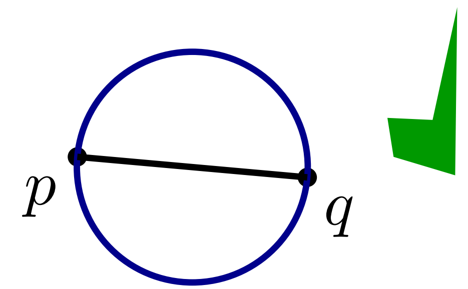

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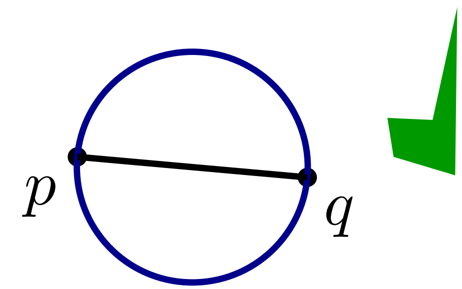

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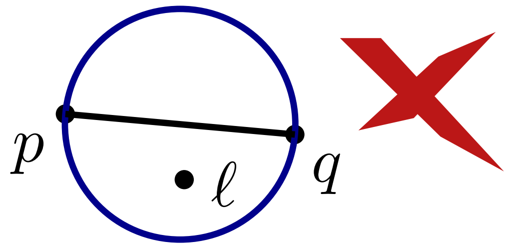


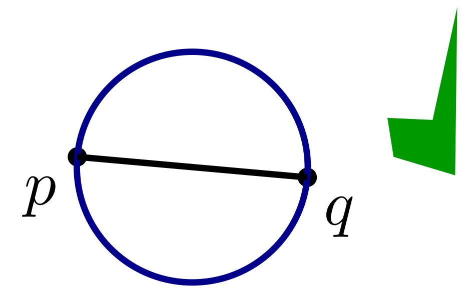

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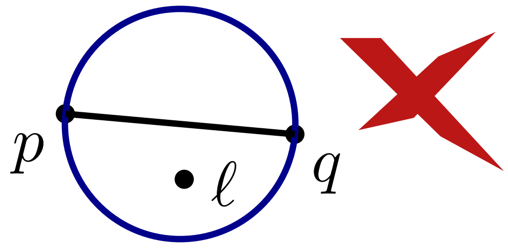
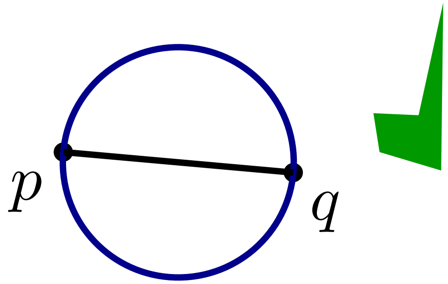


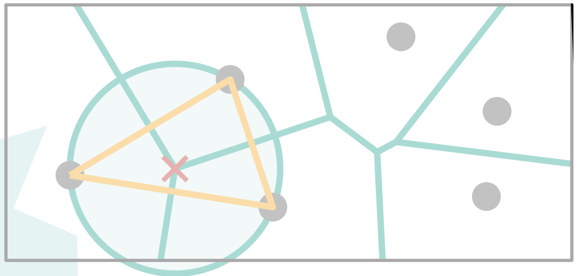



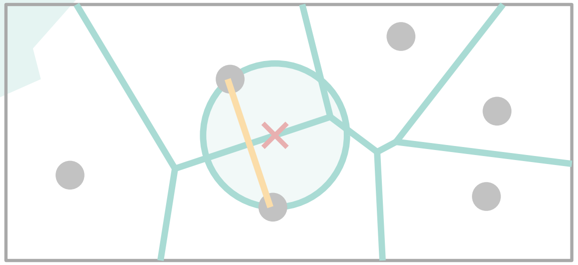

a) Show that Delaunay triangulation of $P$ contains Gabriel graph of $P$.

# Characterization

Theorem about Voronoi-Diagram:

- point $q$ is a Voronoy-vertex
  $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- bisector $b(p_i, p_j)$ defines a Voronoi-edge
  $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.





**Theorem 4:** Let $P$ be a set of points.

- Points $p, q, r$ are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow$
  circle through $p, q, r$ is empty
- Edge $pq$ is in $\mathcal{DG}(P) \Leftrightarrow$
  there is an empty circle $C_{p,q}$ through $p$ and $q$

**Theorem 5:** Let $P$ be a set of points and let $\mathcal{T}$ be a triangulation of $P$. $\mathcal{T}$ is Delaunay-Triangulation $\Leftrightarrow$ the circumcircle of each triangle has an empty interior.

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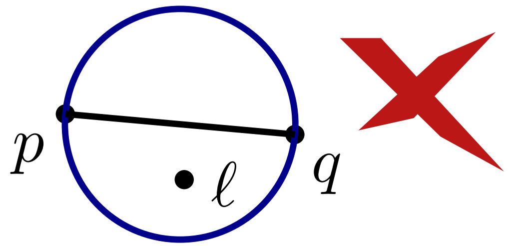


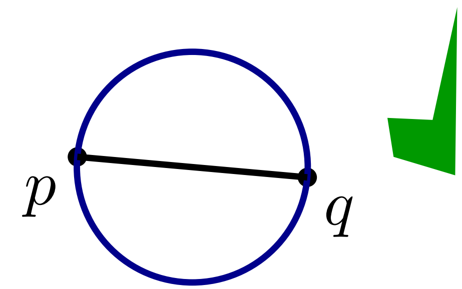


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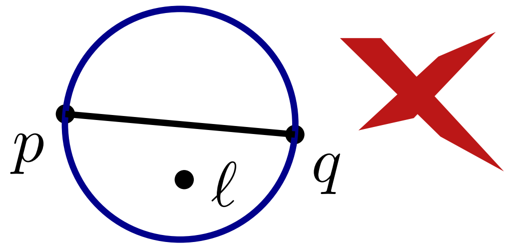


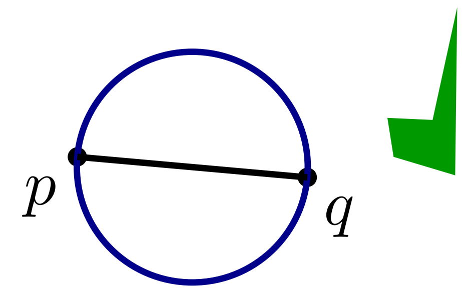


a) Show that Delaunay triangulation of $P$ contains Gabriel graph of $P$.

b) Prove that $p$ and $q$ are adjacent in the Gabriel graph of $P$ **iff** the Delaunay edge between $p$ and $q$ intersects its dual Voronoi edge.

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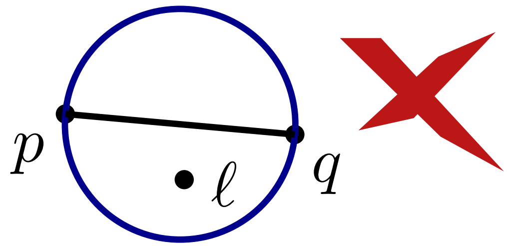


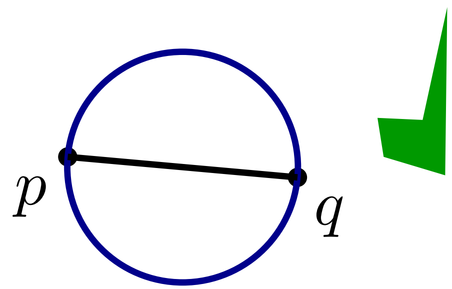


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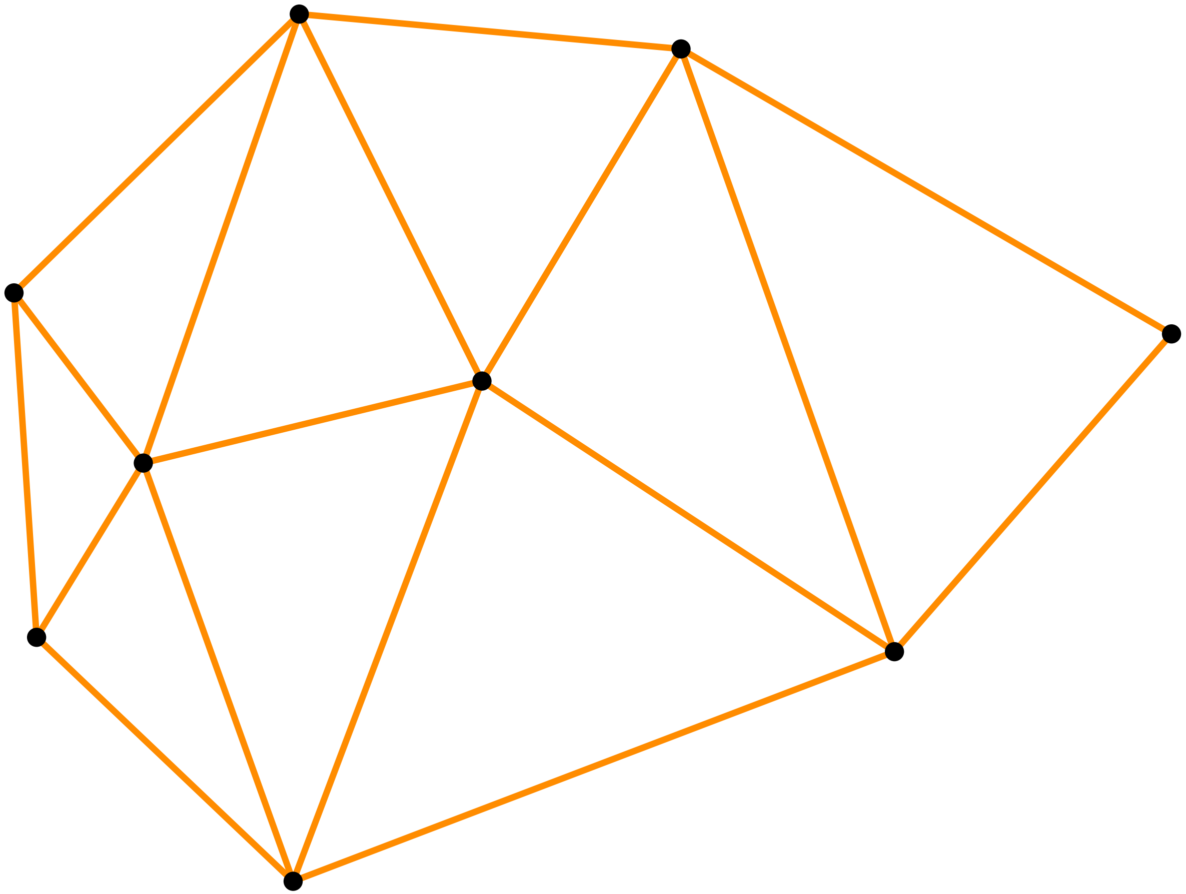

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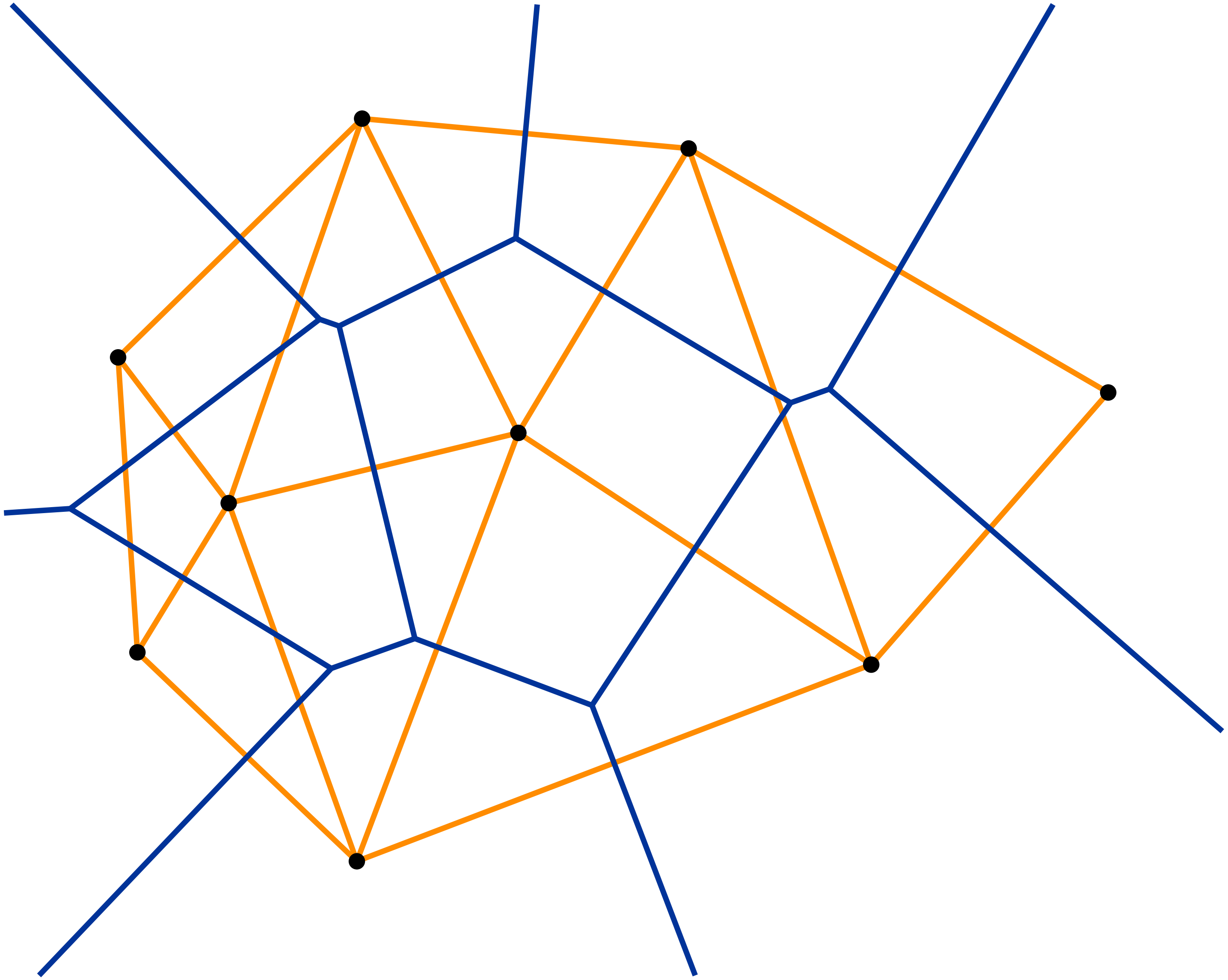

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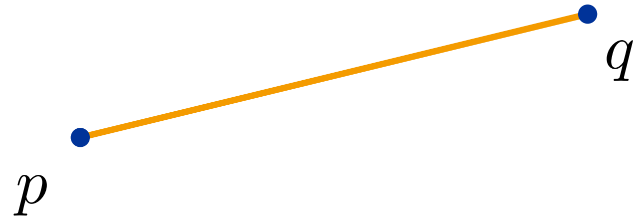

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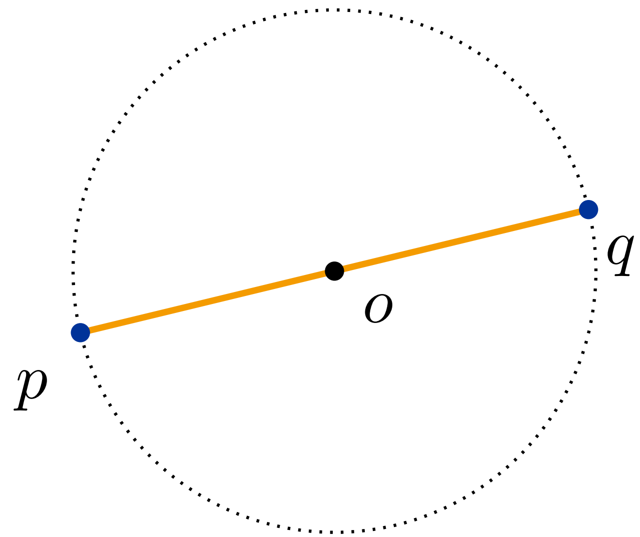

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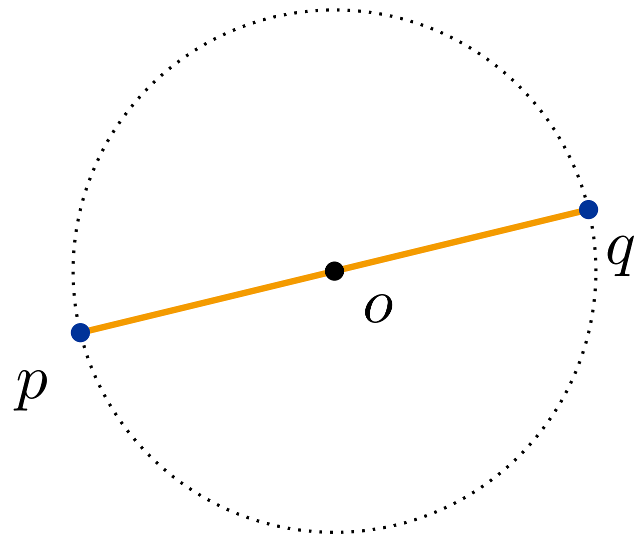


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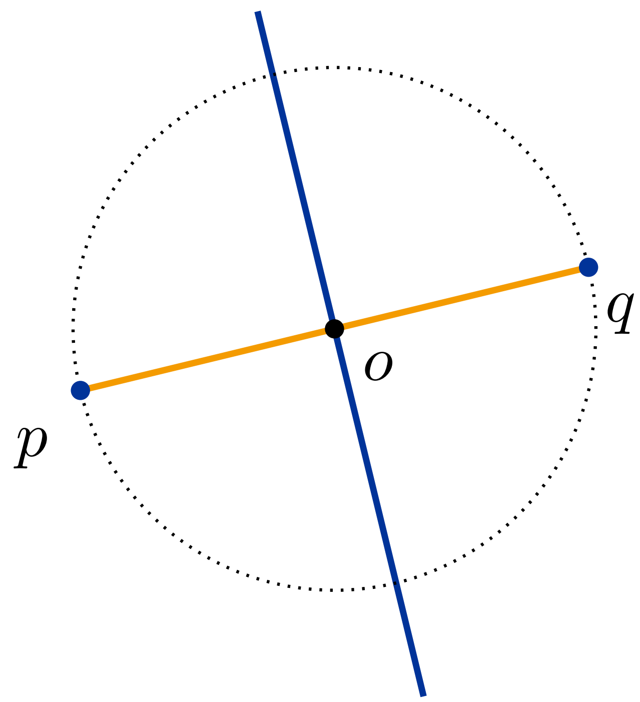


- $o$ is closer to $p$, $q$ than to all other points in $P$
  $\Rightarrow$ $o$ lies on Voronoi-edge which bounds Voronoi cells of $p$ and $q$.

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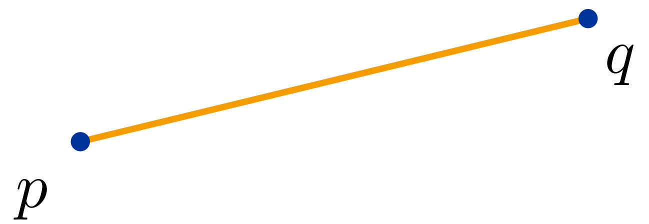
- $pq$ intersects dual Voronoi-edge



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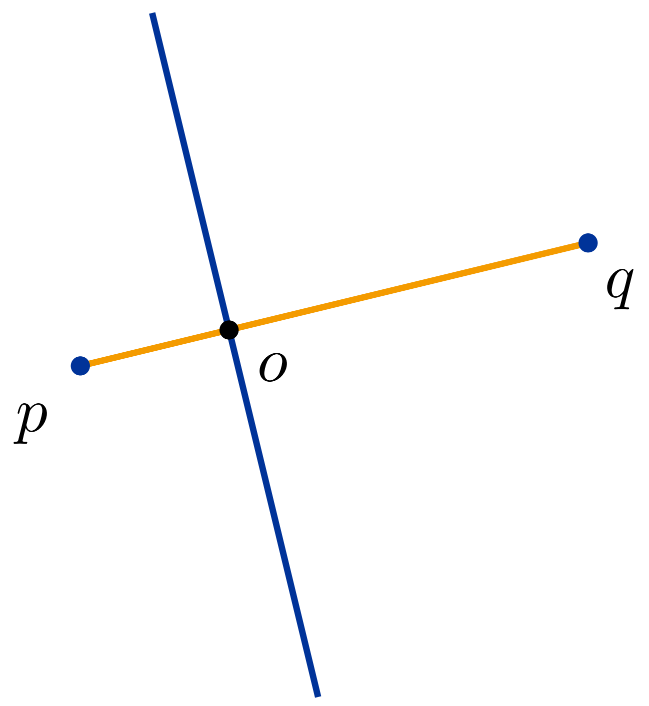

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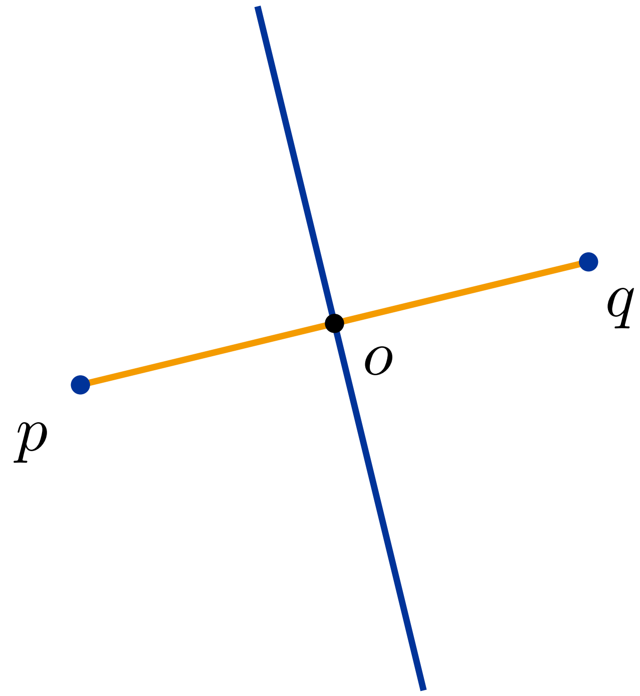

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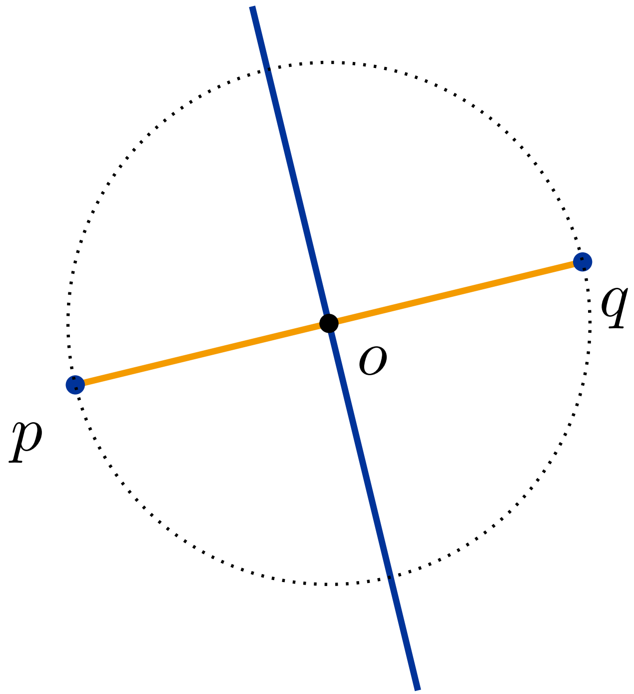

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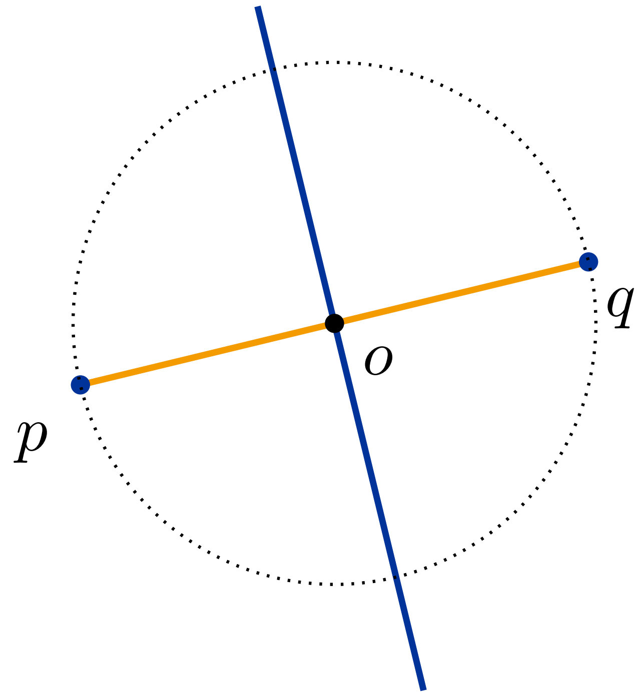


- Assume there is point $r$ in $C$.

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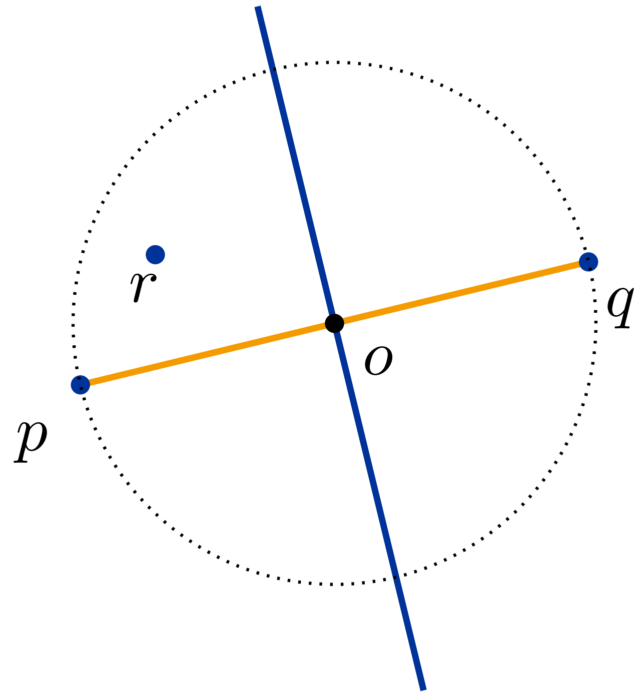


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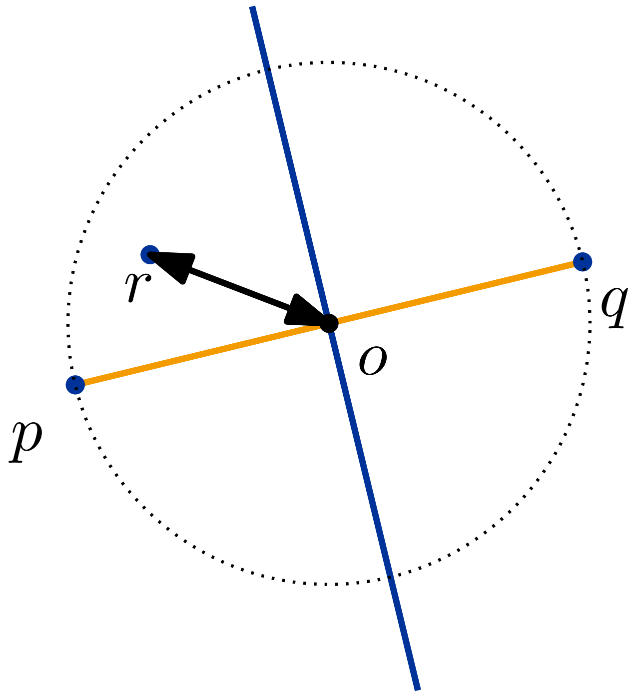


- Assume there is point $r$ in $C$.

  $\Rightarrow$ Distance between $o$ and $r$ less than $|op|$ ($|oq|$)

  $\Rightarrow$ Contradicts assumption that $o$ lies on Voronoi-edge.

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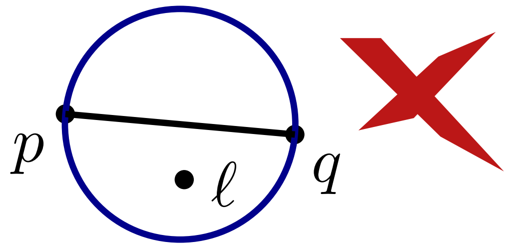


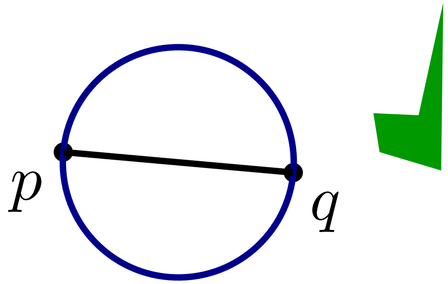


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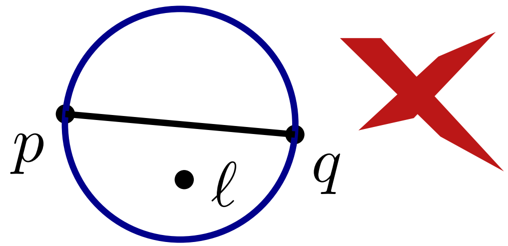


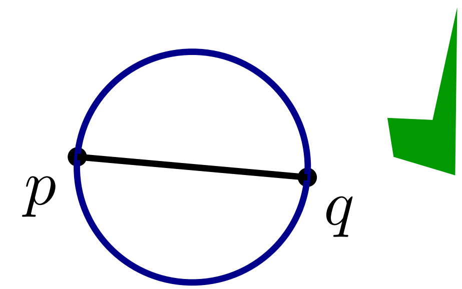


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c) $\mathcal{O}(n \log n)$ Algorithm for computing Gabriel-Graph.