Computational Geometry • Lecture

Well-Separated Pair Decompositions

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

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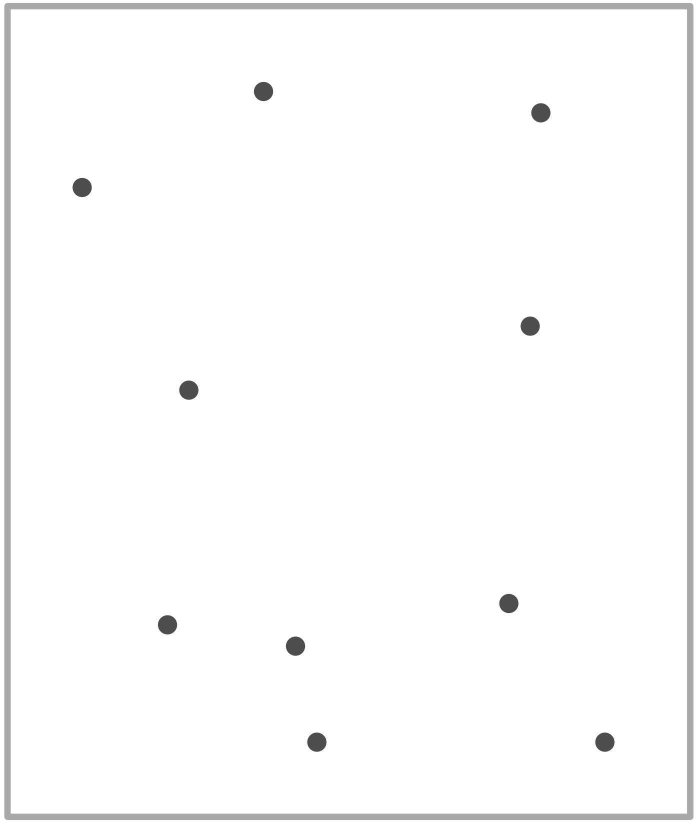
18.1.2016

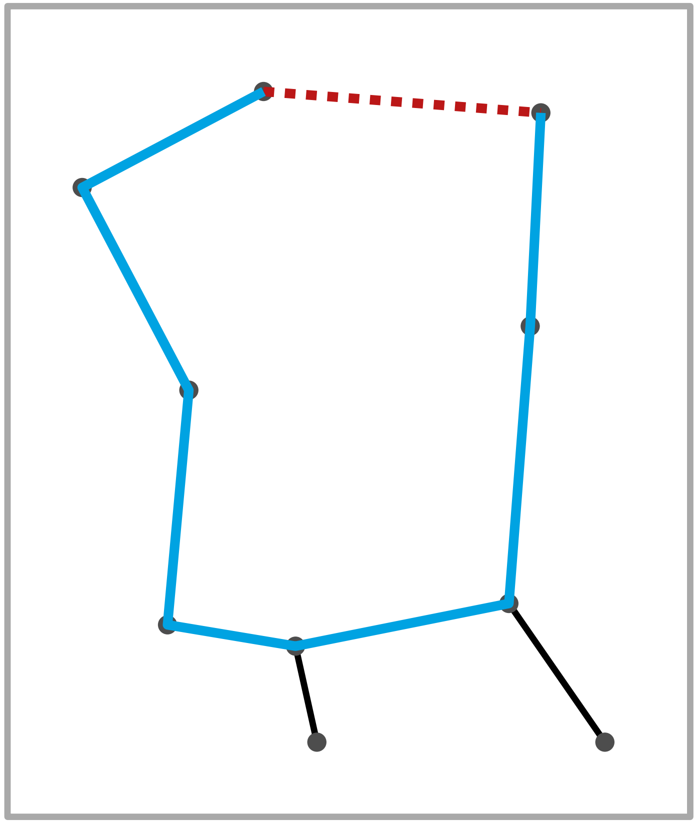
Motivation: Spanners



Task:

A set of cities shall be connected by a new road network.

Motivation: Spanners



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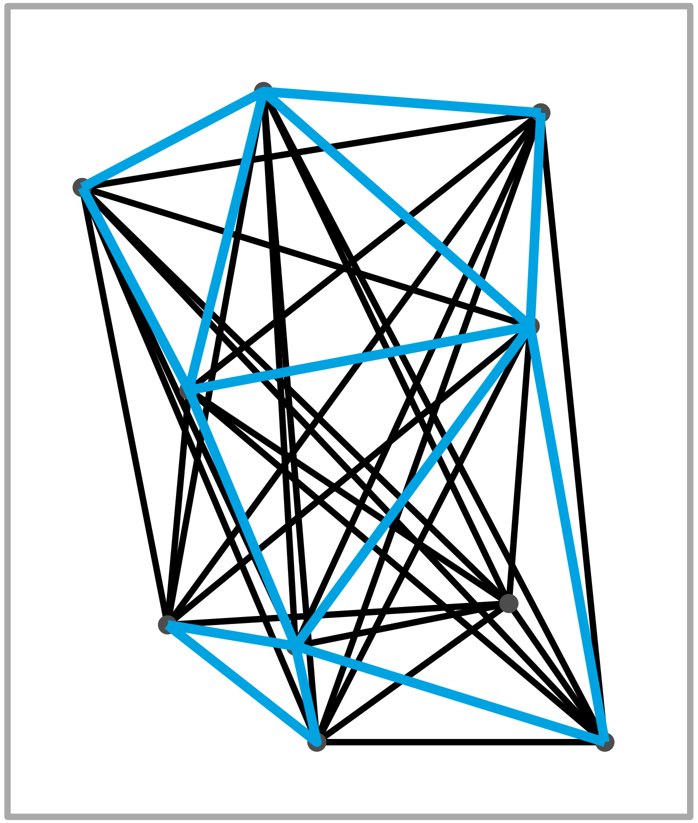
A set of cities shall be connected by a new road network.

But for no pair (x, y) the path length in the road network should be much larger than the distance $||xy||$.

Idea 1: Euclidean minimum spanning tree

Motivation: Spanners



Task:

A set of cities shall be connected by a new road network.

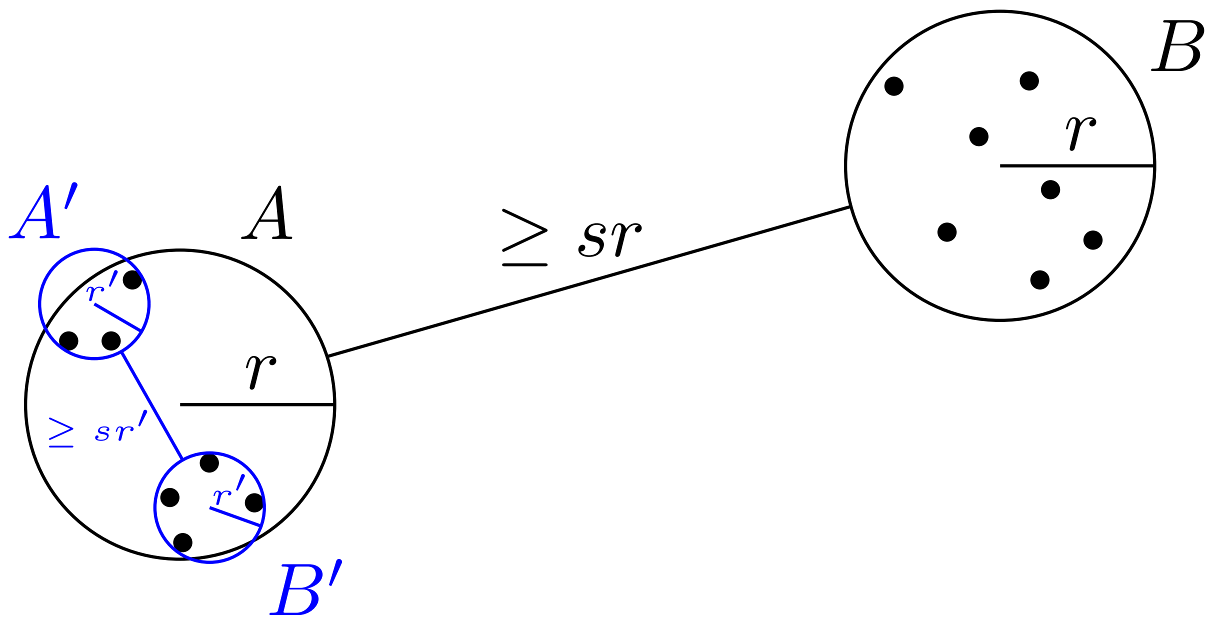
But for no pair (x, y) the path length in the road network should be much larger than the distance $||xy||$.

Construction costs must remain reasonable, e.g., only $O(n)$ edges.

Idea 1: Euclidean minimum spanning tree
Idea 2: complete graph
Idea 3: sparse t-spanner

Well-Separated Pairs

Def: A pair of disjoint point sets A and B in $\mathbb{R}^d$ is called s-**well separated** for some $s > 0$, if A and B can each be covered by a ball of radius r whose distance is at least sr.



Obs:

- s-well separated $\Rightarrow$ s'-well separated for all $s' \leq s$
- singletons $\{a\}$ and $\{b\}$ are s-well separated for all $s > 0$

Well-Separated Pair Decomposition (WSPD)

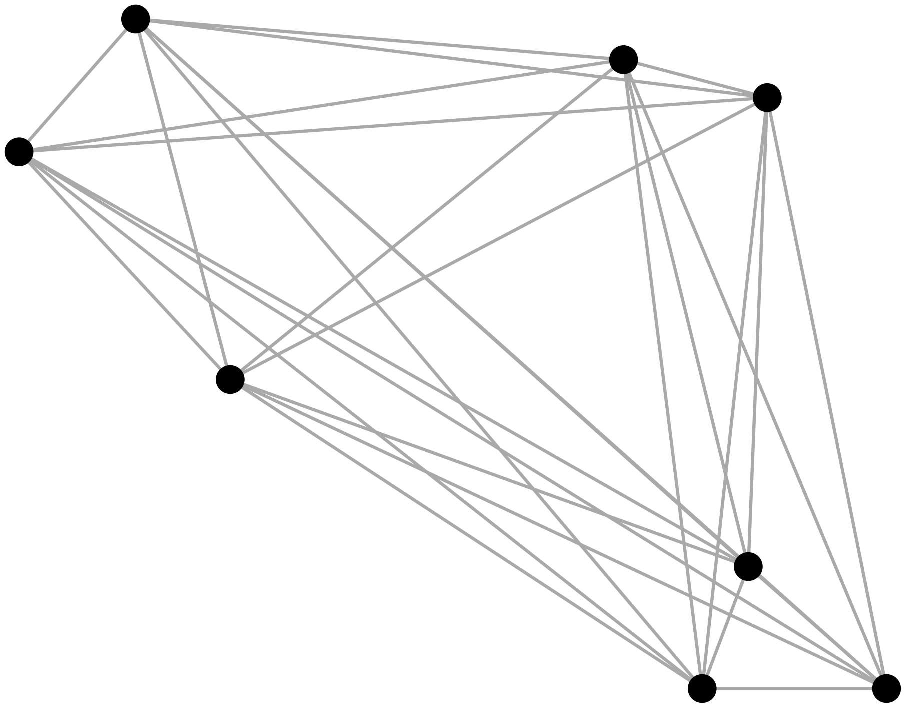
For well-separated pair $\{A, B\}$ we know that the distance for all point pairs in $A \otimes B = \{\{a, b\} \mid a \in A, b \in B, a \neq b\}$ is similar.

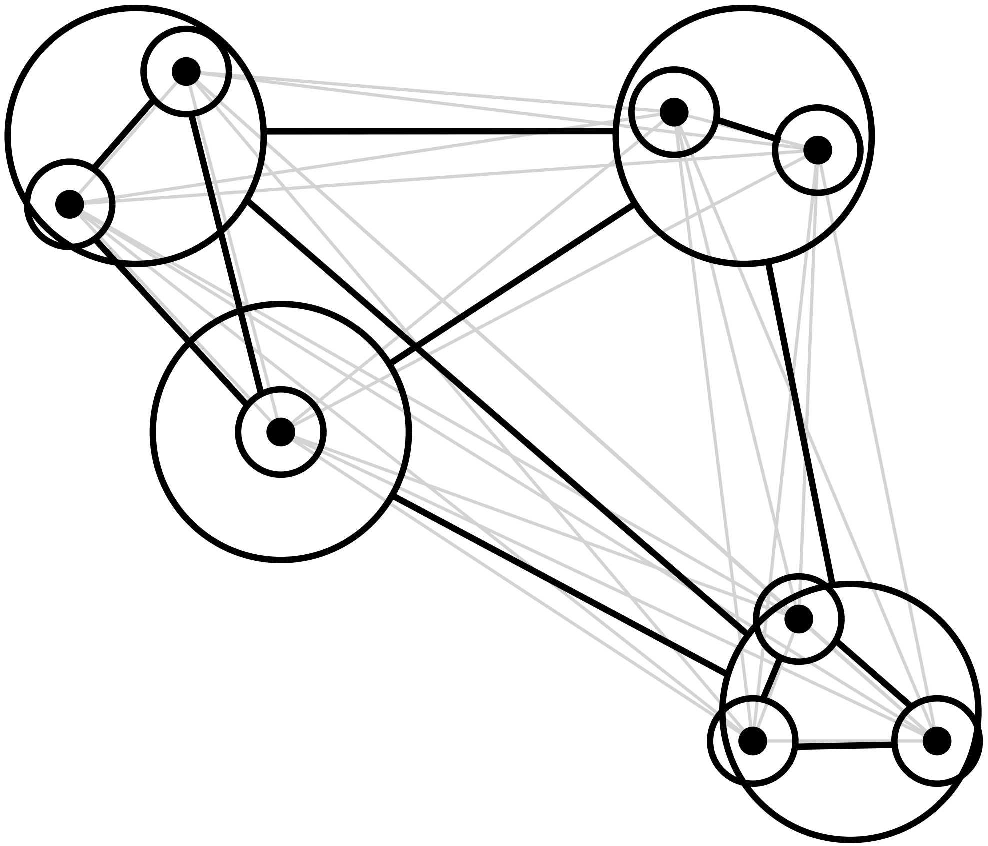
Goal: $o(n^2)$-sized data structure that approximates the distances of all $\binom{n}{2}$ pairs of points in a set $P = \{p_1, \ldots, p_n\}$.

Def: For a point set P and some $s > 0$ an s-**well separated pair decomposition** (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $A_i \cap B_i = \emptyset$ for all i
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ s-well separated for all i

Example





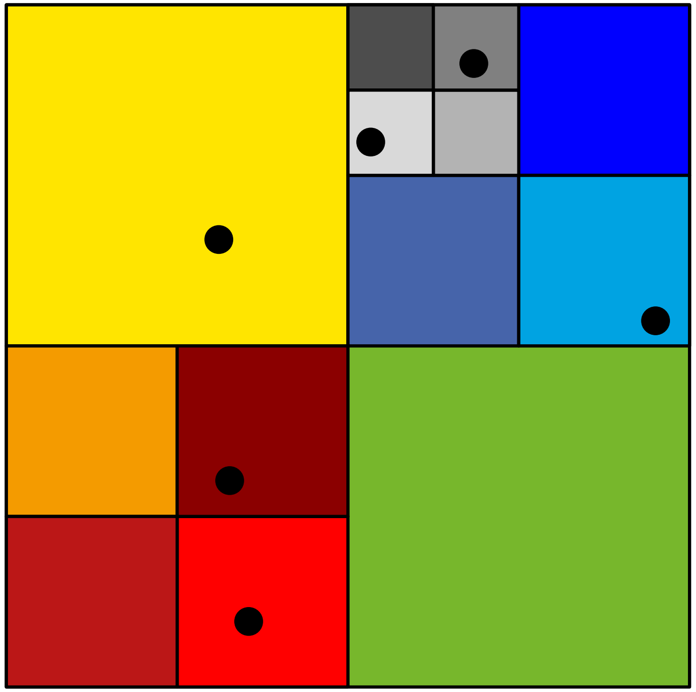
28 point pairs

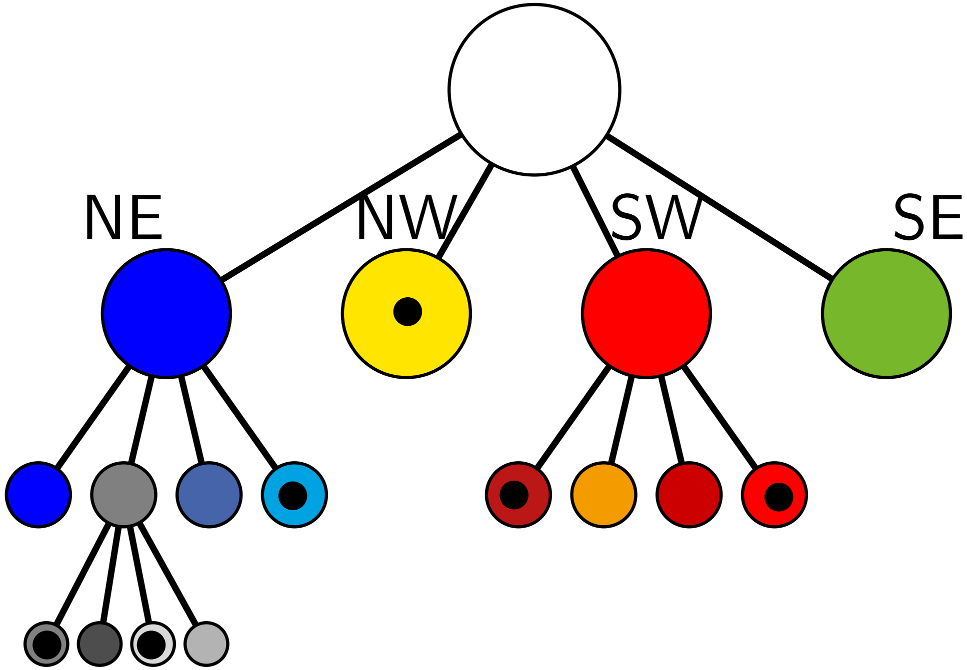
12 s-well separated pairs

WSPD of size $O(n^2)$ is trivial. Can we do it in $O(n)$?

Recall: Quadtrees

Def: A **quadtree** $\mathcal{T}(P)$ for a point set P is a rooted tree, where each internal node has four children. Each node corresponds to a square, and the squares of the leaves form a partition of the root square.





Recall: Quadtrees

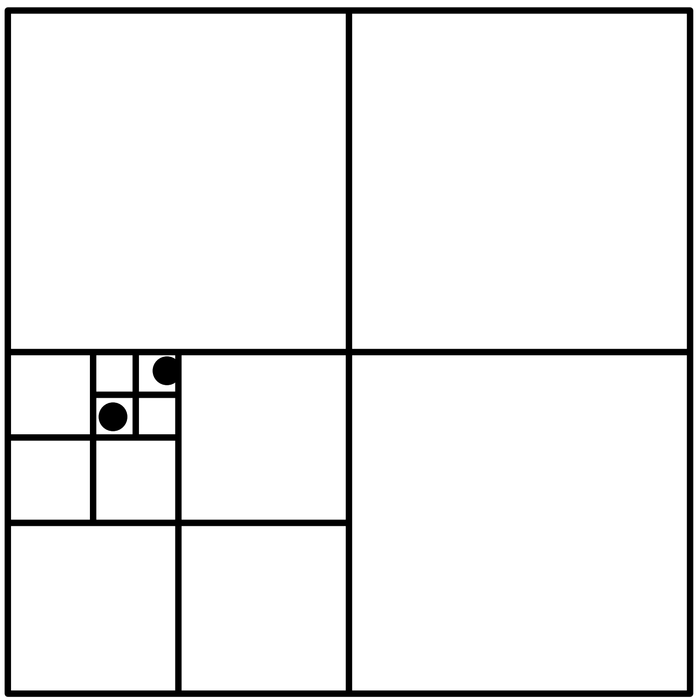
Def: A **quadtree** $\mathcal{T}(P)$ for a point set P is a rooted tree, where each internal node has four children. Each node corresponds to a square, and the squares of the leaves form a partition of the root square.

Lemma 1: The height of $\mathcal{T}(P)$ is at most $\log(s/c) + 3/2$, where c is the smallest distance in P and s is the side length of the root square Q.

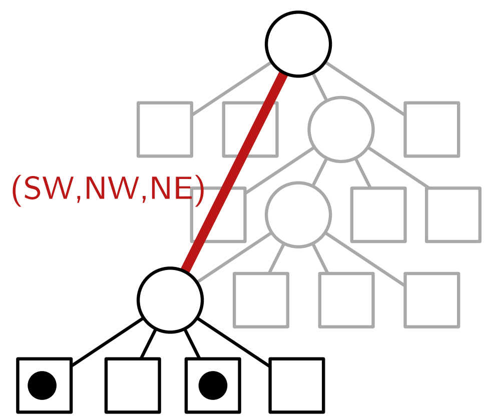
Thm 1: A quadtree $\mathcal{T}(P)$ on n points with height h has $O(hn)$ nodes and can be constructed in $O(hn)$ time.

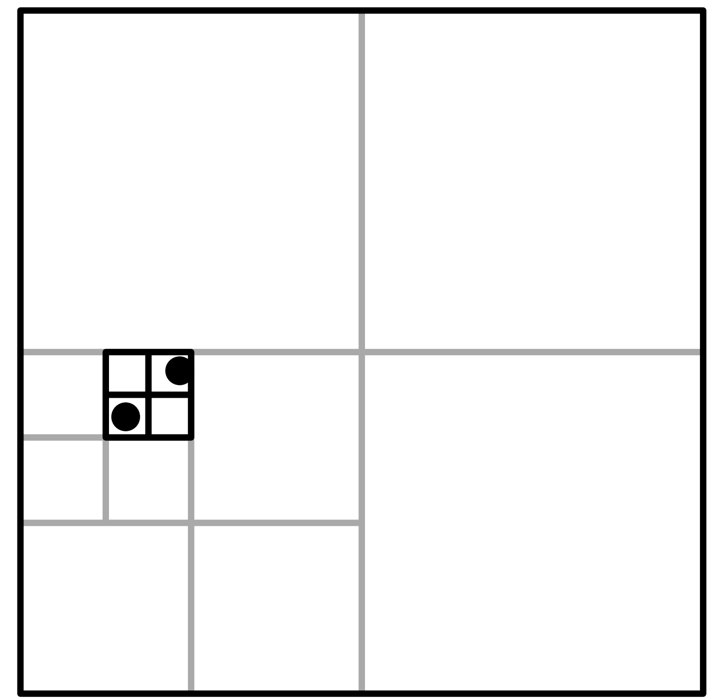
Compressed Quadtrees

Def: A **compressed** quadtree is a quadtree, in which each path of non-separating inner nodes is contracted into a single edge. Each such edge has a label to reconstruct the path structure.



quadtree





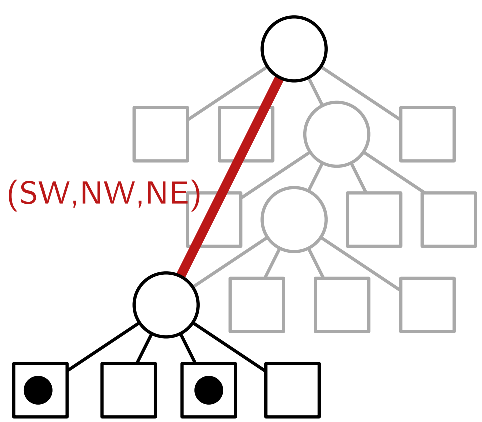
compressed quadtree

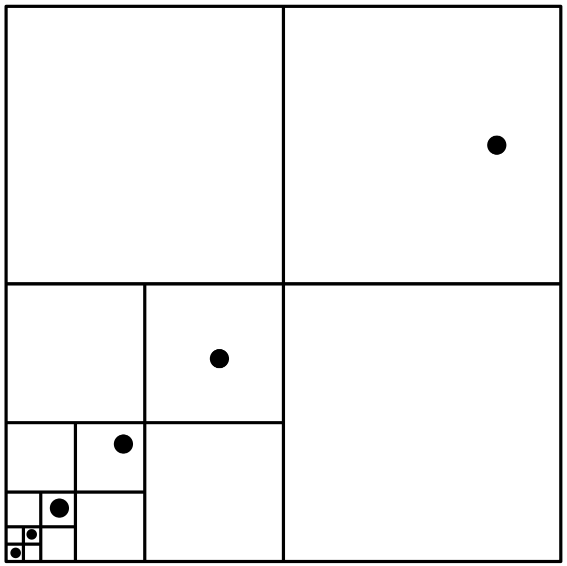
Properties of Compressed Quadtrees

Obs:
- inner nodes split their point set into ≥ 2 non-empty parts $\Rightarrow$ max. $n-1$ inner nodes
- depth can be $d = n$, so the algorithm to construct quadtrees takes $O(n^2)$ time

Thm 2: A compressed quadtree for n points in $\mathbb{R}^d$ with a fixed dimension d can be constructed in $O(n \log n)$ time.

e.g. skip-quadtree [Eppstein et al. 2005] (without proof)



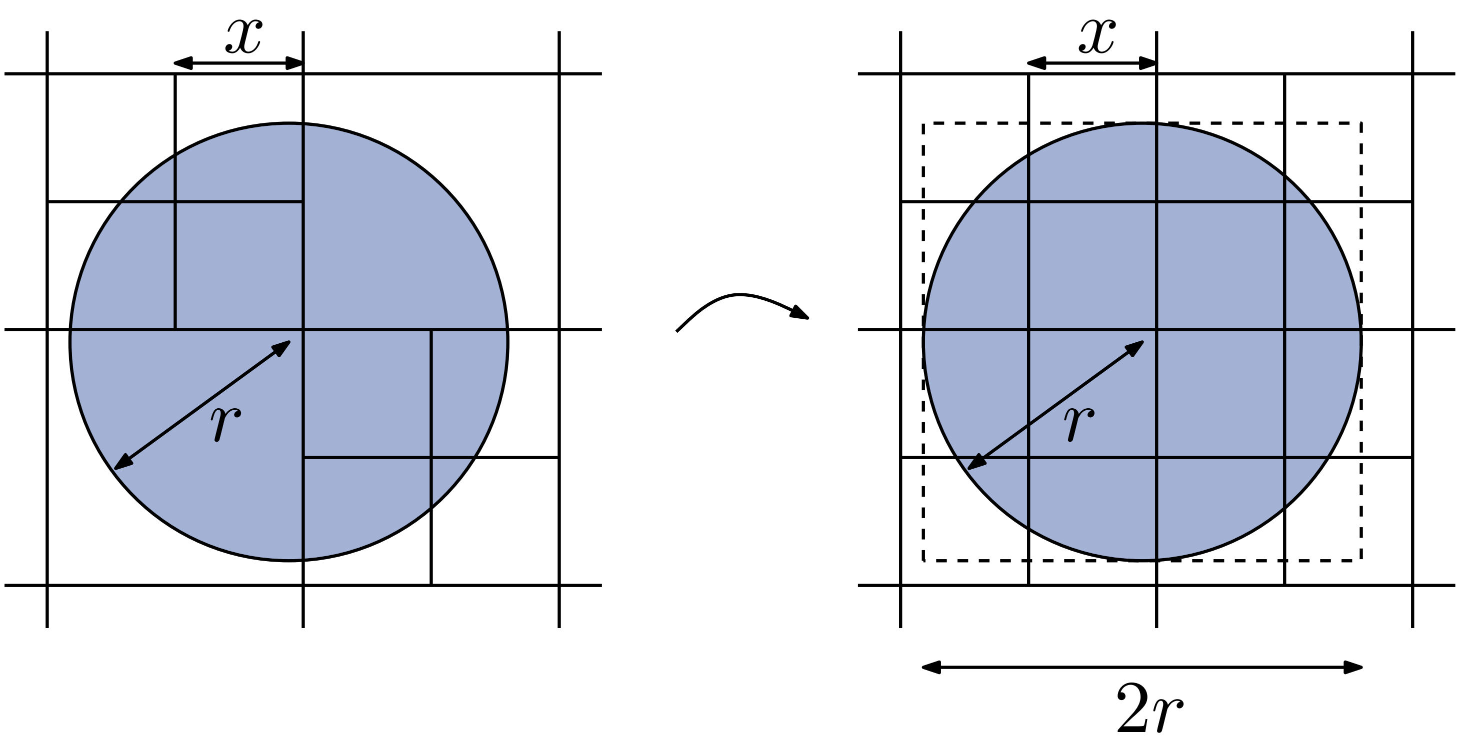


Packing Lemma

Lemma 2: Let K be a ball with radius r in $\mathbb{R}^d$ and let X be a set of pairwise disjoint quadtree cells with side length $\geq x$ that intersect K. Then it holds

$$|X| \leq (1 + \lceil 2r/x \rceil)^d.$$

Proof:



Representatives and Level

Def: For each node u of a quadtree $\mathcal{T}(P)$ for point set P let $P_u = Q_u \cap P$ be the set of points in the corresponding square Q_u. In each leaf u define the representative

$$\text{rep}(u) = \begin{cases} p & \text{falls } P_u = \{p\} \ (u \text{ is leaf}) \\ \emptyset & \text{otherwise.} \end{cases}$$

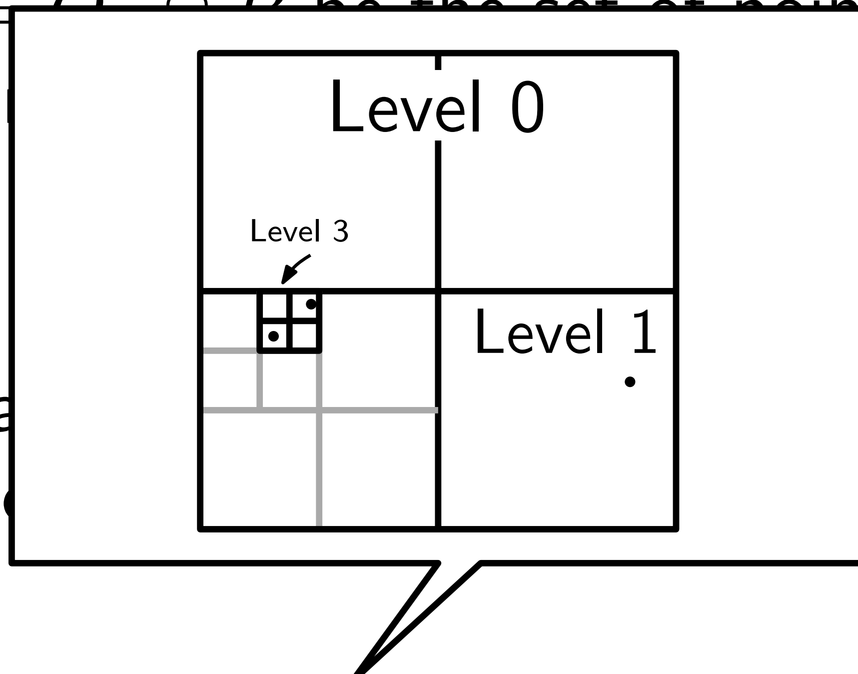
For an inner node v assign $\text{rep}(v) = \text{rep}(u)$ for a non-empty child u of v.

Def: For each node u of a quadtree $\mathcal{T}(P)$ let $\text{level}(u)$ be the level of u in the corresponding *uncompressed* quadtree. We have $\text{level}(u) \leq \text{level}(v)$ iff $\text{area}(Q_u) \geq \text{area}(Q_v)$.

Representatives and Level

Def: For each node u of a quadtree $\mathcal{T}(P)$ for point set P let P_u = ... ts in the corresponding squa... the representative ...} (u is leaf)

For a...) = rep(u) for a non-...

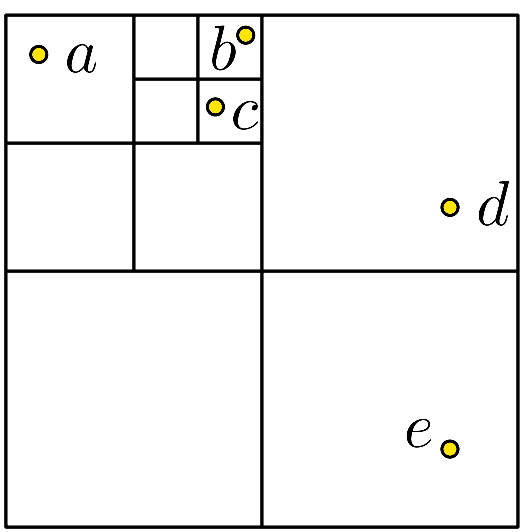


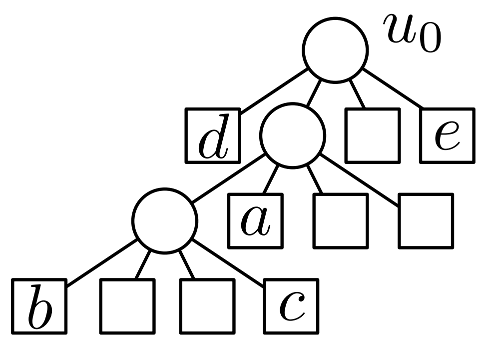
Def: For each node u of a quadtree $\mathcal{T}(P)$ let level(u) be the level of u in the corresponding *uncompressed* quadtree. We have level(u) $\leq$ level(v) iff area(Q_u) $\geq$ area(Q_v).

Constructing a WSPD

wsPairs($u, v, \mathcal{T}, s$)

Input: quadtree nodes u, v, quadtree $\mathcal{T}$, $s > 0$
Output: WSPD for $P_u \otimes P_v$
if rep(u) $= \emptyset$ or rep(v) $= \emptyset$ or leaf $u = v$ **then return** $\emptyset$
else if P_u and P_v s-well separated **then return** $\{\{u, v\}\}$
else
    if level(u) $>$ level(v) **then** swap u and v
    $(u_1, \ldots, u_m) \leftarrow$ children of u in $\mathcal{T}$
    return $\bigcup_{i=1}^{m}$ wsPairs($u_i, v, \mathcal{T}, s$)





Constructing a WSPD

wsPairs($u, v, \mathcal{T}, s$)

Input: quadt

Output: WS

circles around Q_u and Q_v (or radius 0 for point in a leaf)

increase smaller circle and check if distance $\geq sr$

if rep(u) $= \emptyset$ or rep(v) $= \emptyset$ or leaf $u = v$ **then return** $\emptyset$

else if P_u and P_v s-well separated **then return** $\{\{u, v\}\}$

else

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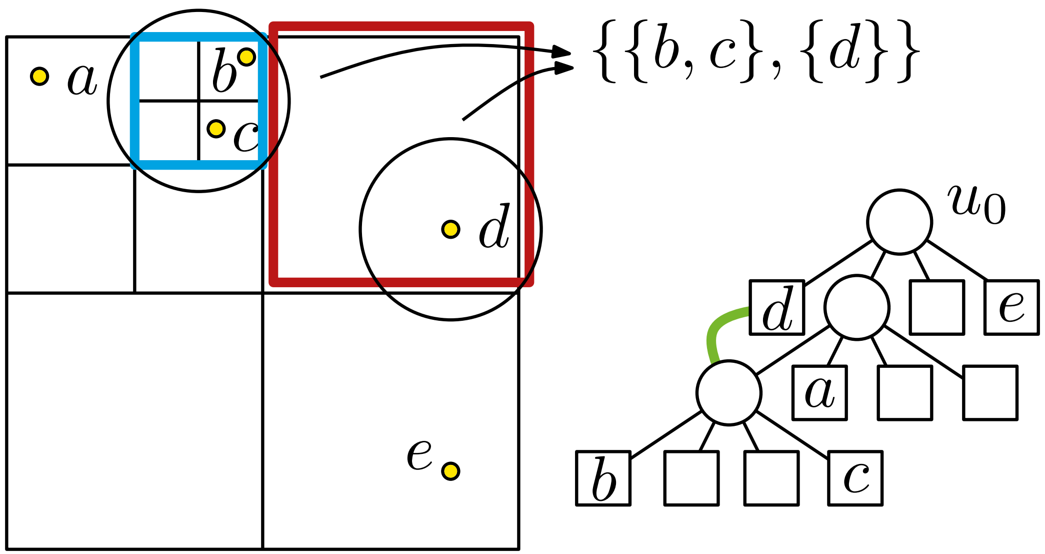
$\{\{b, c\}, \{d\}\}$



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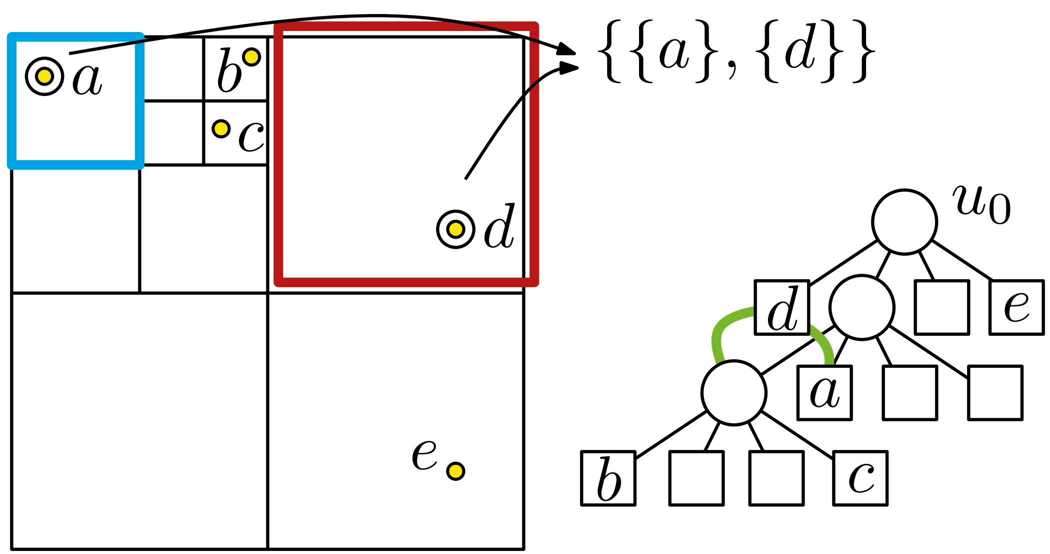

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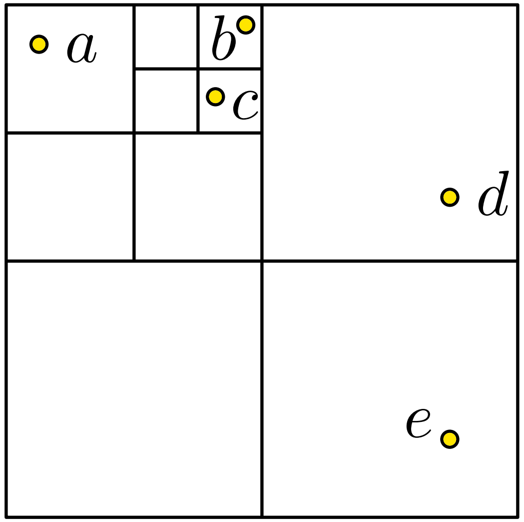

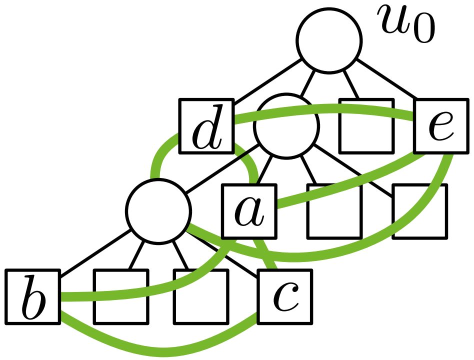

$\{\{b, c\}, \{d\}\}$
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Constructing a WSPD

Karlsruhe Institute of Technology

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- initial call wsPairs$(u_0, u_0, \mathcal{T}, s)$
- avoid duplicates wsPairs$(u_i, u_j, \mathcal{T}, s)$ and wsPairs$(u_j, u_i, \mathcal{T}, s)$ How?
- leaf pairs are always s-well separated, so algorithm terminates
- output are pairs of quadtree nodes

Space use?

Question: How many pairs does the algorithm create?

Analysis of WSPD Construction

Thm 3: Given a point set P in $\mathbb{R}^d$ and $s \geq 1$ we can construct an s-WSPD with $O(s^d n)$ pairs in time $O(n \log n + s^d n)$.

Sketch of proof:

- simplifying assumption: no quadtree compression required
  $\Rightarrow$ in wsPairs$(u, v, \mathcal{T}, s)$ sizes of u and v differ by at most factor 2
- **goal:** count calls to wsPairs
  - call is **trivial** if it produces no further recursive calls
  - each trivial call produces at most one ws pair
  - each non-trivial call produces $\leq 2^d$ trivial calls and thus $\leq 2^d$ ws pairs
- let's count non-trivial calls and charge cost to the smaller of the two cells
- call non-trivial $\Rightarrow$ u and v not ws, $u \geq v$
- let x be side length of v and $r_v = x\sqrt{d}/2$ the radius of the enclosing ball
- side length of u is x or $2x$ and $r_u \leq 2r_v$
- u, v not ws $\Rightarrow$ ball distance $\leq s \max\{r_u, r_v\} \leq 2sr_v = sx\sqrt{d}$

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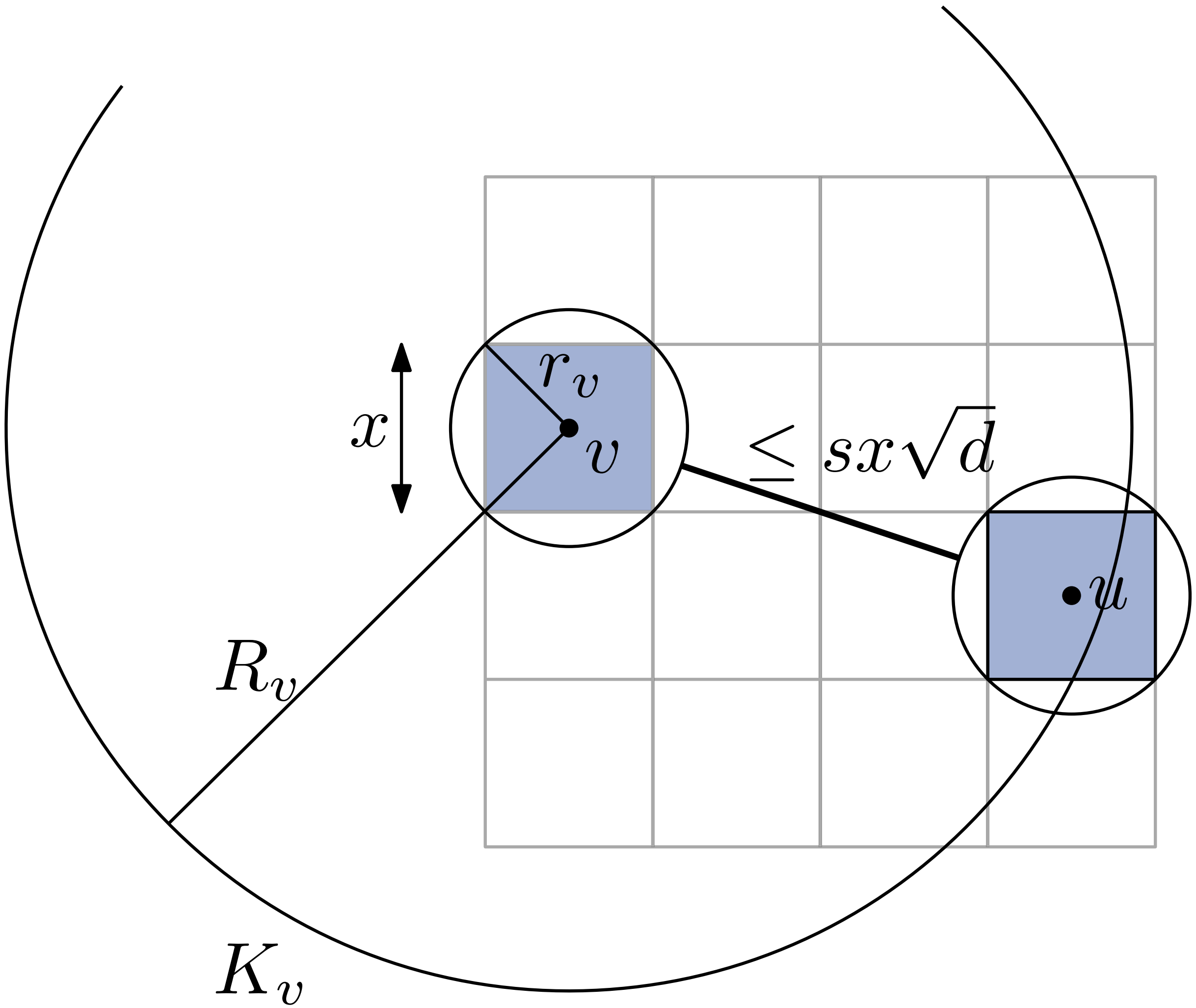

Recall Lemma 2:
Given ball K with radius r in $\mathbb{R}^d$ and set X of pairwise disjoint quadtree cells with side length $\geq x$ that intersect K. Then

$$|X| \leq (1 + \lceil 2r/x \rceil)^d.$$

- all cells charging cost to v have size x or $2x$ and intersect K_v; let C be their number and apply Lemma 2 (see board)
- yields $C = O(s^d)$

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Sketch of proof:

- have $O(n)$ nodes in $\mathcal{T}$
- each causes $O(s^d)$ non-trivial calls
- each non-trivial call produces $O(2^d)$ ws-pairs

- in total $O(s^d n)$ ws-pairs
- time: $O(n \log n)$ for quadtree and $O(s^d n)$ for the s-WSPD □

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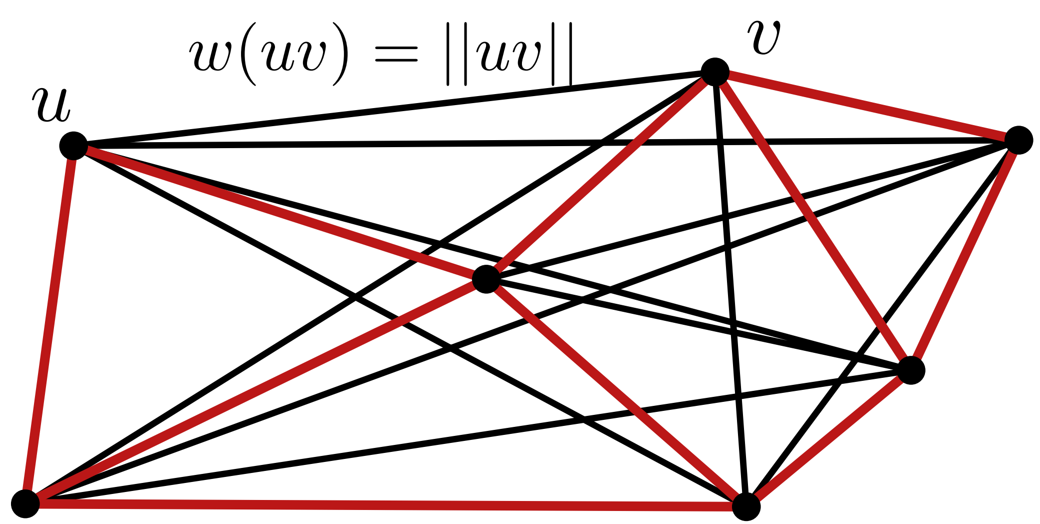
t-Spanner

For a set P of n points in $\mathbb{R}^d$ the **Euclidean graph** $\mathcal{EG}(P) = (P, \binom{P}{2})$ is the complete weighted graph, whose edge weights correspond to the Euclidean distances of the edges' endpoints.

Since $\mathcal{EG}(P)$ has $\Theta(n^2)$ edges, one is often interested in a sparse graphs with $O(n)$ edges, whose shortest paths approximate the edge weights in $\mathcal{EG}(P)$.



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Def: A weighted graph G with vertex set P is called t-**spanner** for P and a stretch factor $t \geq 1$, if for all pairs $x, y \in P$ it holds

$$||xy|| \leq \delta_G(x, y) \leq t \cdot ||xy||,$$

where $\delta_G(x, y) =$ length of shortest x-y-path in G.

WSPD und t-Spanner

Def: For n points P in $\mathbb{R}^d$ and a WSPD W of P define the graph $G = (P, E)$, where
$E = \{\{x, y\} \mid \exists\{u, v\} \in W \text{ with } \mathrm{rep}(u) = x, \mathrm{rep}(v) = y\}$.

Recall: For each node u of a quadtree $\mathcal{T}(P)$ for point set P let $P_u = Q_u \cap P$ be the set of points in the corresponding square Q_u.
In each leaf u define the representative

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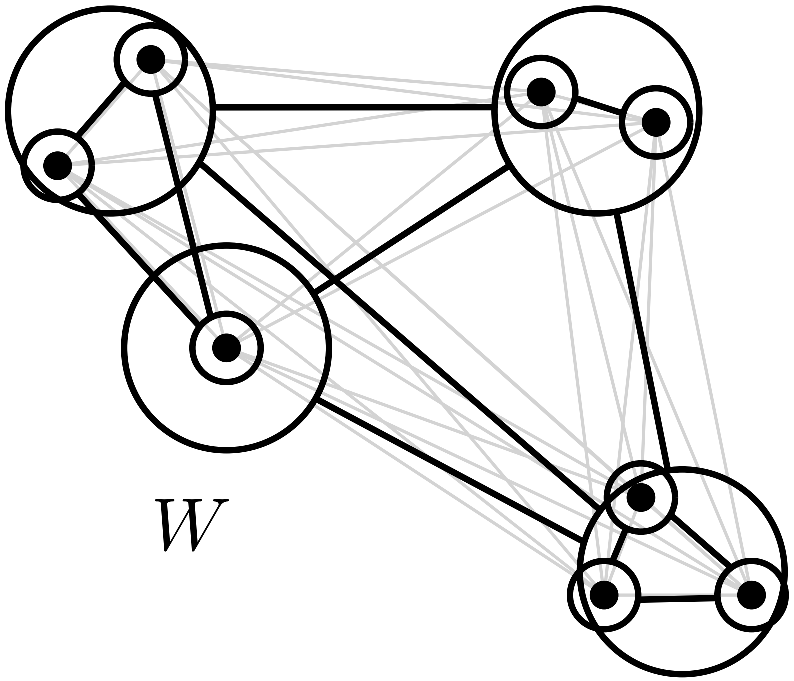

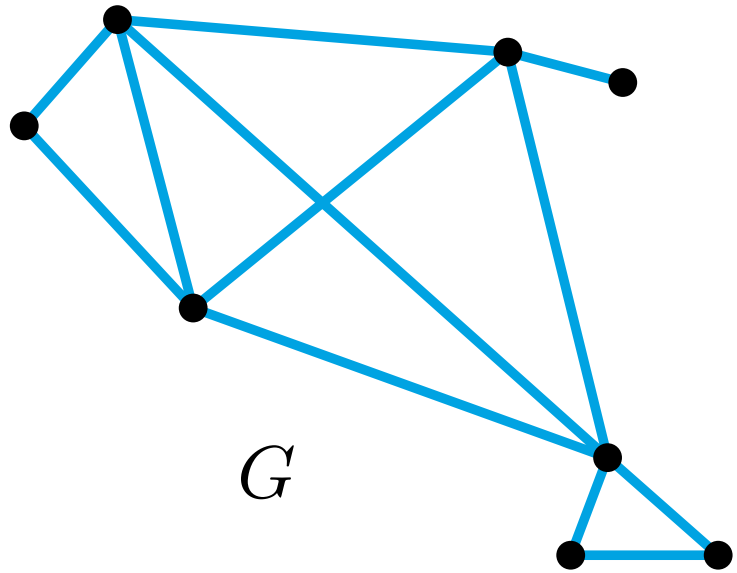

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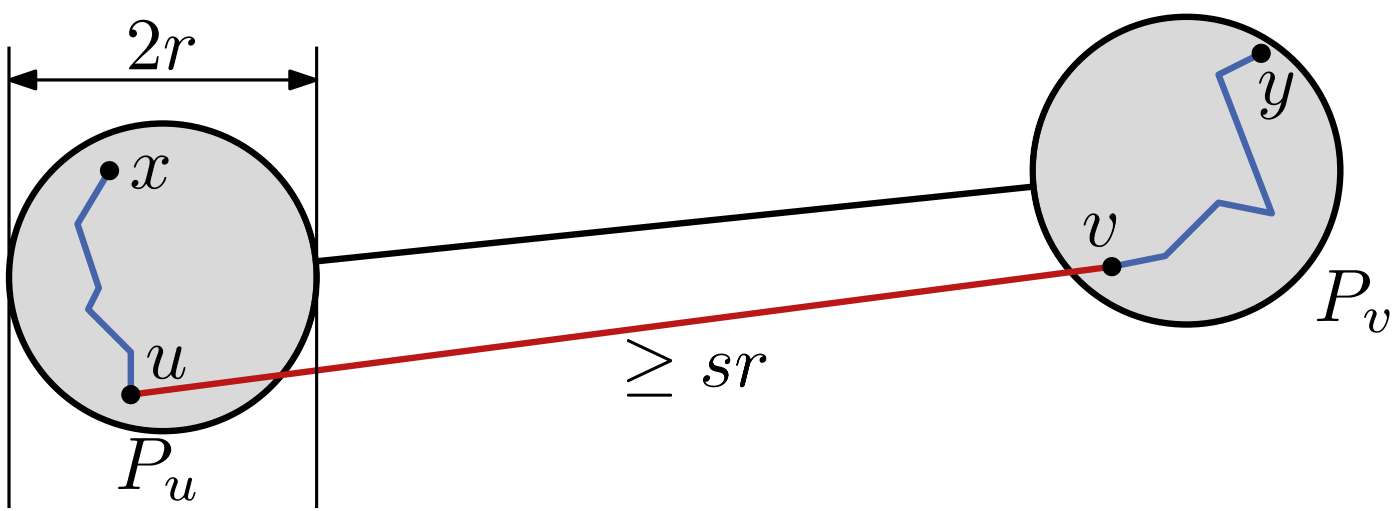
Lemma 3: If W is a s-WSPD for a suitable $s = s(t) \geq 4$, then G is a t-spanner for P with $O(s^d n)$ edges.

Proof: (blackboard)



Summary

Thm 4: For a set P of n points in $\mathbb{R}^d$ and some $\varepsilon \in (0,1]$ we can compute an $(1+\varepsilon)$-spanner for P with $O(n/\varepsilon^d)$ edges in $O(n \log n + n/\varepsilon^d)$ time.

Proof: For $t = (1+\varepsilon)$ we have with $s = 4 \cdot \frac{t+1}{t-1}$ that

$$O(s^d n) = O\left(\left(4 \cdot \frac{2+\varepsilon}{\varepsilon}\right)^d n\right) \subseteq O\left(\left(\frac{12}{\varepsilon}\right)^d n\right) = O\left(\frac{n}{\varepsilon^d}\right)$$

P

↓ $O(n \log n)$

compressed quadtree

↓ $O(n/\varepsilon^d)$

WSPD

↓ $O(n/\varepsilon^d)$

$(1+\varepsilon)$-spanner

□