Algorithms for graph visualization
Layouts for planar graphs. Realizer method.
Barycentric Representation

Let $A, B, C, P \in \mathbb{R}^2$. A triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that:

- $\alpha + \beta + \gamma = 1$
- $P = \alpha A + \beta B + \gamma C$

is called barycentric coordinates of $P$ with respect to $\triangle ABC$. 
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A Barycentric Representation of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of $G$, i.e. it is an injective function $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$, such that:
- $v_a + v_b + v_c = 1$ for all $v \in V$
- for each $(x, y) \in E$ and each $z \in V \setminus \{x, y\}$, $\exists k \in \{a, b, c\}$ with $x_k < z_k$ and $y_k < z_k$. 
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What does this condition mean?
Barycentric Representation

Lemma [Schnyder ’90]

Let \( v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3 \) be a barycentric representation of a graph \( G = (V, E) \) and let \( A, B, C \in \mathbb{R}^2 \). The function

\[
f: v \in V \mapsto v_a A + v_b B + v_c C
\]

gives a planar drawing of \( G \) inside triangle \( \triangle ABC \).
Definition: Schnyder-Labeling

A Schnyder-Labeling of a planar triangulated graph $G$ is a labeling of all internal angles with labels 1, 2 and 3 such that:

- **Face**: Each internal face contains vertices with all three labels 1, 2, and 3, appearing in a counterclockwise order.
- **Vertex**: The counterclockwise ordering of the labels around each vertex consists of a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.
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Schnyder Labeling

Theorem [Schnyder ’90]

Every triangulated plane graph has a Schnyder labeling.
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- Edge contraction. Contractible edge. Notation: \( G \setminus (u, v) \).
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**Lemma**

Let $G$ be a triangulated plane graph with vertices $a$, $b$, $c$ on the outer face. There exists a contractible edge $(a, x)$ in $G$, $x \neq b, c$. 
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Let \( G \) be a triangulated plane graph with vertices \( a, b, c \) on the outer face. There exists a contractible edge \((a, x)\) in \( G \), \( x \neq b, c \).

Proof

- By induction on the number of vertices in a graph.
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- Assume that every graph with less or equal than \( k - 1 \) vertices has a Schnyder labeling in which all labels at \( a \) are 1.
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Schnyder Labeling & Forest

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Definition: Schnyder Forest

A Schnyder Forest or a Realizer of a planar triangulated graph $G = (V, E)$ is a partition of the inner edges of $E$ into three sets of oriented edges $T_1, T_2, T_3$ such that for each inner vertex $v \in V$ hold:

- $v$ has an outgoing edge in each of $T_1, T_2, T_3$
- The counterclockwise order of the edges around $v$ is as follows: edges leaving in $T_1$, entering in $T_3$, leaving in $T_2$, entering in $T_1$, leaving in $T_3$, entering in $T_2$. 
Schnyder Forest

Recall that:

Theorem [Schnyder ’90]

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By this theorem and by previous construction:

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- For each $v$ there exists a directed red, blue, green paths from $v$ to $a, b, c$, respectively.
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- No monochromatix cycle exists
- Each monochromatic subgraph is a tree!
- The sinks of red/blue/green trees are the vertices $a, b, c$. 

Schnyder Forest
Face Regions

- Paths $P_a(v)$, $P_b(v)$, $P_c(v)$ cross only at vertex $v$.
- $R_a(v)$, $R_b(v)$, $R_c(v)$ are sets of faces.
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For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.
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Proof ...
Barycentric Representation

Let barycentric coordinates of $v \in G \setminus \{a, b, c\}$ be $(v_a, v_b, v_c)$, where $v_a = |R_a(v)|/(2n - 5)$, $v_b = |R_b(v)|/(2n - 5)$ and $v_c = |R_c(v)|/(2n - 5)$.

We set: $A = (2n - 5, 0)$, $B = (0, 2n - 5)$, $C = (0, 0)$. 
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**Satz [Schnyder ’90]**

The function

$$f : v \mapsto (v_a, v_b, v_c) = \frac{1}{2n - 5}(|R_a(v)|, |R_b(v)|, |R_c(v)|)$$

is a barycentric representation of $G$. 
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- Condition1: \( v_a + v_b + v_c = 1 \).
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Proof

- **Condition 1**: \( v_a + v_b + v_c = 1 \).

- **Condition 2**: For each edge \((u, v)\) and vertex \( w \neq u, v\) at least one of three is true: \( w_a > u_a, v_a\), \( w_b > u_b, v_b\), \( w_c > u_c, v_c\).
Final Remarks

- The resulting drawing is a grid drawing.
- It is bounded by the triangle $\triangle ABC$ with $A = (2n - 5, 0)$, $B = (0, 2n - 5)$, $C = (0, 0)$.
- It has area $2n - 5 \times 2n - 5$. 
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How to obtain area $n - 2 \times n - 2$?

- Use weak barycentric coordinates $\frac{1}{n-1}(n_1(v), n_2(v), n_3(v))$, $n_i(v) = |\text{vertices in } R_i(v)| - |P_{i-1}(v)|$ with respect to $A = (n - 1, 0)$, $B = (0, n - 1)$, $C = (0, 0)$.
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Weak barycentric coordinates: Triple $(v_a, v_b, v_c)$ such that

- $v_a + v_b + v_c = 1$
- For each edge $(u, v)$ and vertex $w \neq u, v$, $\exists k \in \{a, b, c\}$, such that $(u_k, u_{k+1}) <_{\text{lex}} (w_k, w_{k+1})$, and $(v_k, v_{k+1}) <_{\text{lex}} (w_k, w_{k+1})$.
- Here we say that $(a, b) <_{\text{lex}} (c, d)$ iff $a < c$ or $a = c$ and $b < d$. 