Exercise Sheet 2

Assignment: November 28, 2014
Delivery: None, Discussion on December 8, 2014

1 Generalization of Canonical Ordering for Triconnected Planar Graphs

The canonical ordering (which until now has been defined only for maximal planar graphs) can be generalized to triconnected planar graphs as follows. Let \( G = (V, E) \) be a triconnected planar graph. An ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) is called canonical if vertices \( v_2, v_n \) are neighbors of \( v_1 \), and \( v_1, v_2, v_n \) belong to a common face, and for every \( k \), \( k > 3 \):

(a) Vertex \( v_k \) is on the outer face of \( G_k \) and has at least two neighbors in \( G_{k-1} \), which are on the outer face of \( G_{k-2} \). Vertex \( v_k \) has at least one neighbor in \( G - G_k \). Graph \( G_k \) is biconnected.

(b) or there exists an \( \ell \geq 1 \) such that \( v_k, \ldots, v_{k+\ell} \) is a chain on the outer face of \( G_{k+\ell} \) and has exactly two neighbors in \( G_{k-1} \), which are on the outer face of \( G_{k-2} \). Every vertex \( v_k, \ldots, v_{k+\ell} \) has at least one neighbor in \( G - G_{k+\ell} \). Finally, \( G_{k+\ell} \) is biconnected.

By \( G_k \) we denote the subgraph of \( G \) induced by the vertices \( v_1, \ldots, v_k \) and by \( G - G_k \) the subgraph of \( G \) resulting from \( G \) by removing the vertices and edges of \( G_k \).

Prove that every triconnected planar graph admits a canonical ordering.

2 Constrained Canonical Ordering

Let \( G \) be a maximal plane graph with vertices \( v_1, v_2, v_n \) on the outer face. Let \( P \) be a simple path in \( G \) connecting vertices \( v_1 \) and \( v_2 \) and not containing \( v_n \). Let \( G_P \) be the subgraph of \( G \) bounded by path \( P \) and edge \((v_1, v_2)\). Prove that there exists a canonical ordering of \( G \) such that all the vertices of \( G_P \) appear as initial subsequence of this ordering.

3 Visibility Representation of Maximal Planar Graphs

Recall the definition of visibility representation from the previous exercise set. Prove that every maximal planar graph admits a visibility representation.

Hint: Use canonical ordering.

4 Barycentric Coordinates

Let \( \Delta_{a,b,c} \) be a triangle on the plane on vertices \( a, b \) and \( c \). For each point \( x \) laying inside triangle \( \Delta_{a,b,c} \) there exists a triple \((x_a, x_b, x_c)\) such that \( x_a \cdot a + x_b \cdot b + x_c \cdot c = x \) and \( x_a + x_b + x_c = 1 \). The triple \((x_a, x_b, x_c)\) is called barycentric coordinates of \( x \) with respect to \( \Delta_{a,b,c} \).
Prove that:

(a) If \( A(\Delta) \) denotes the area of the triangle \( \Delta \), then

\[
x_a = \frac{A(\Delta_{b,c,x})}{A(\Delta_{a,b,c})}, \quad x_b = \frac{A(\Delta_{a,c,x})}{A(\Delta_{a,b,c})}, \quad x_c = \frac{A(\Delta_{a,b,x})}{A(\Delta_{a,b,c})}
\]

(b) Equations \( x_a = 0, x_b = 0, x_c = 0 \) represent lines through \( bc, ab \) and \( ab \), respectively.

(c) Let \((x_a, x_b, x_c)\) be barycentric coordinates of point \( x \) in triangle \( \Delta_{abc} \). The set of points \( \{(x_a, x'_b, x'_c) : x'_b, x'_c \in \mathbb{R}\} \) represents a line parallel to edge \( bc \) passing through point \( x \).

Similarly, sets of points \( \{(x'_a, x_b, x'_c) : x'_a, x'_c \in \mathbb{R}\} \) and \( \{(x'_a, x'_b, x_c) : x'_a, x'_b \in \mathbb{R}\} \) represent lines parallel to edges \( ac, ab \), respectively, passing through point \( x \).

5 Linear time construction of a Schnyder realizer.

Let \( G \) be a maximal planar graph with \( n \) vertices. Can a Schnyder labeling and a Schnyder realizer be constructed in time \( O(n) \)?

**Hint:** Find a Connection between a canonical ordering and the ordering in which the edge contraction for the construction of a Schnyder labeling is executed.

6 Induced path in a Schnyder realizer.

A path of a graph \( G \) is called *induced* if the vertices of this path are connected only by the edges of the path, i.e. path on vertices \( v_1, \ldots, v_k \) is induced if for any \( 1 \leq i, j \leq n \) such that \( |i - j| > 1 \), edge \((v_i, v_j)\) does not belong to \( G \). Let \( G \) be a maximal planar graph and let \( T_a, T_b, T_c \) be a Schnyder realizer of \( G \). Assume that the edges of \( T_a, T_b, T_c \) are colored red, blue and green, respectively. Show that a directed monochromatic path in \( T_a, T_b, T_c \) is an induced path of \( G \).