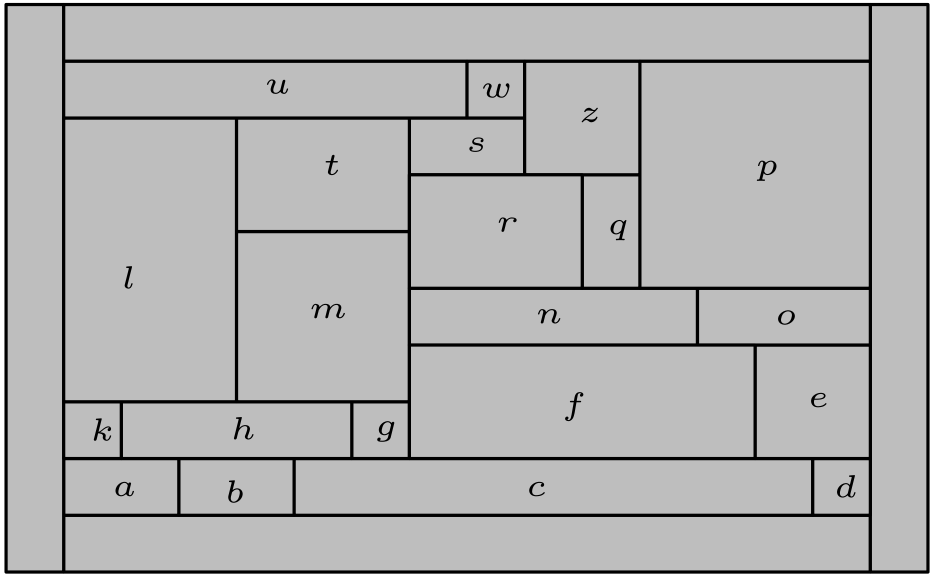
Algorithms for graph visualization

Contact representations of planar graphs.

Winter semester 2013/2014

Tamara Mchedlidze – Martin Nöllenburg



Contact representation

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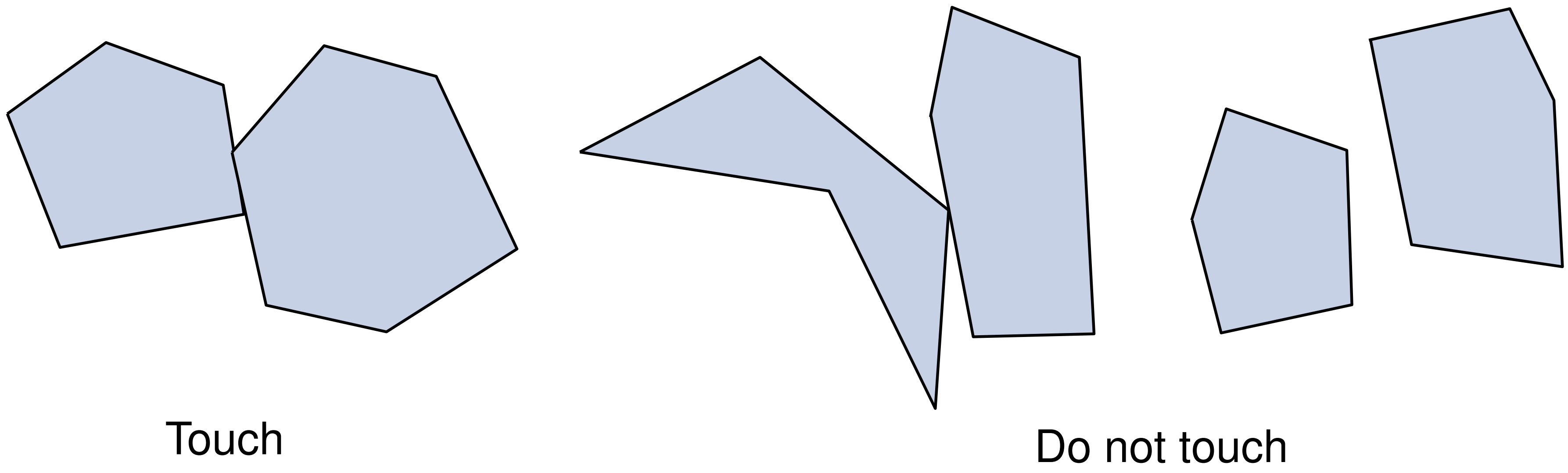
In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.



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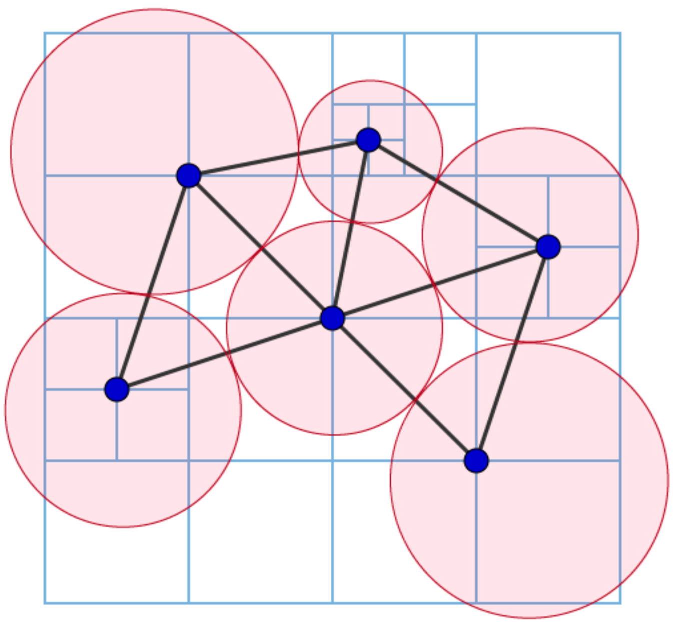

Touching disk representation

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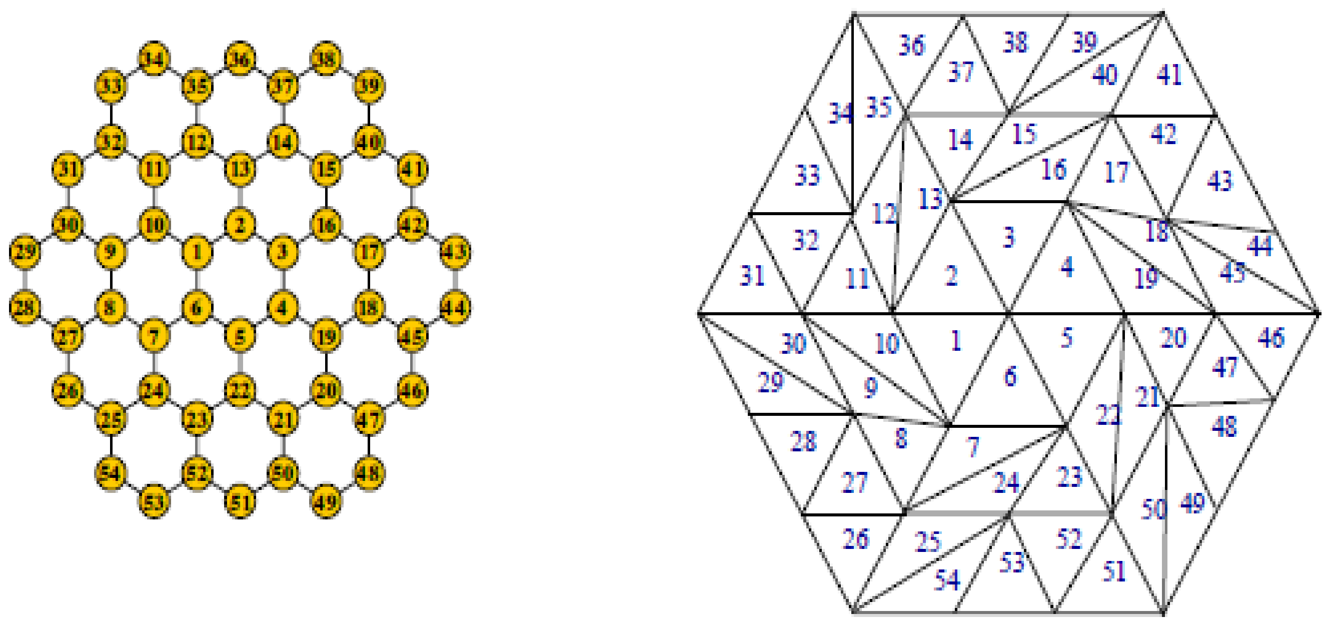

Touching triangle representation

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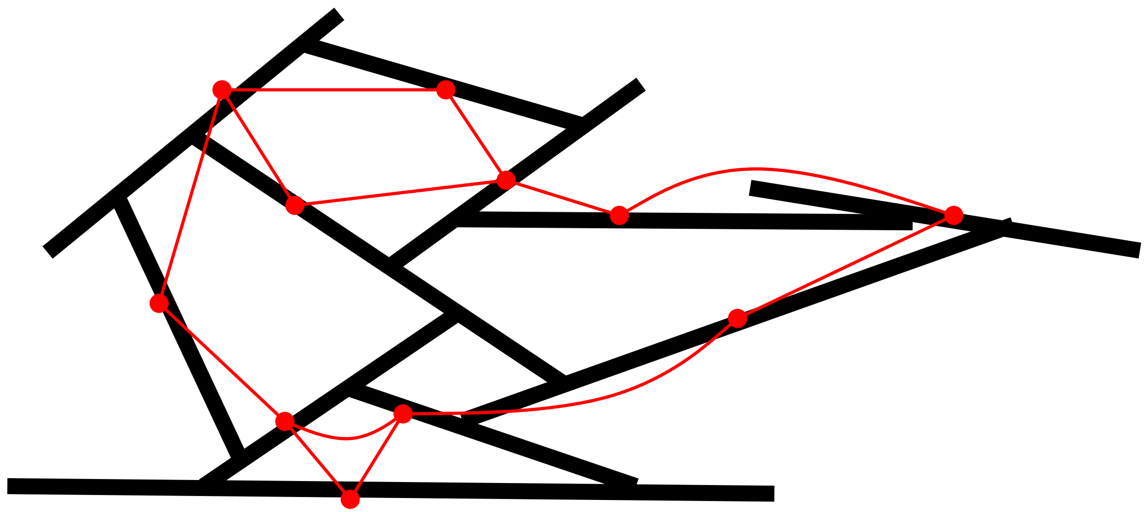

Touching segment representation

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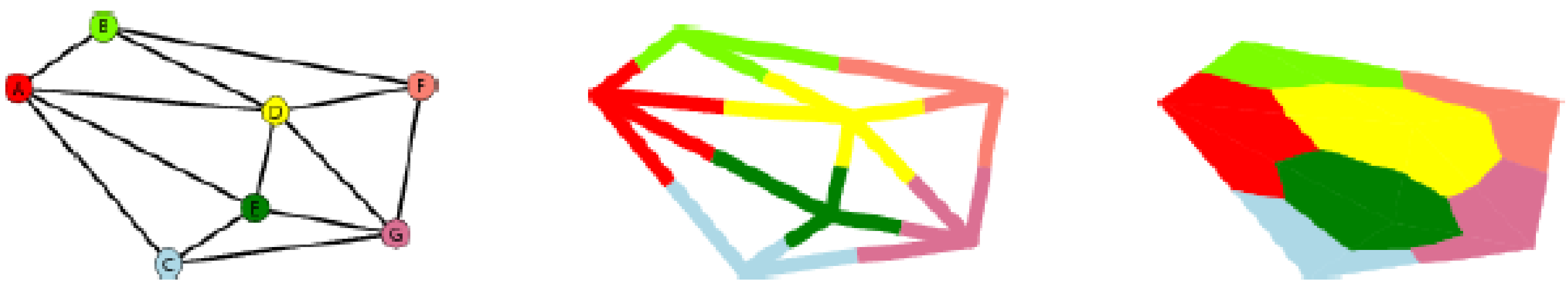

General idea for the construction of a contact representation of a planar graph using n-gons in worst case

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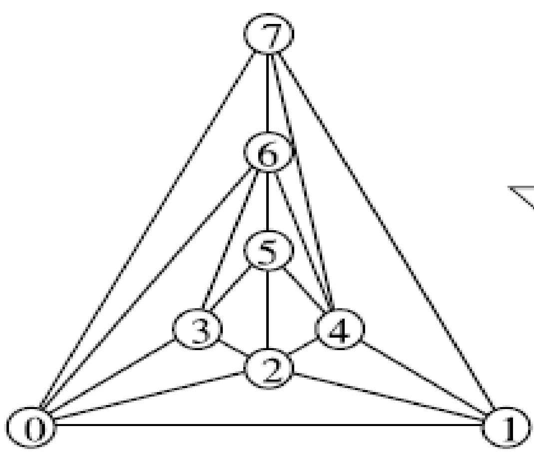

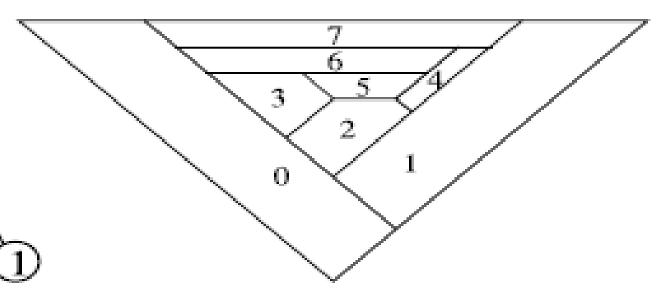

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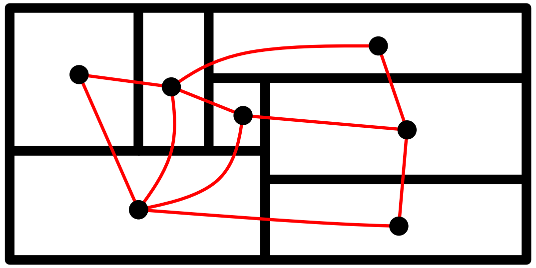

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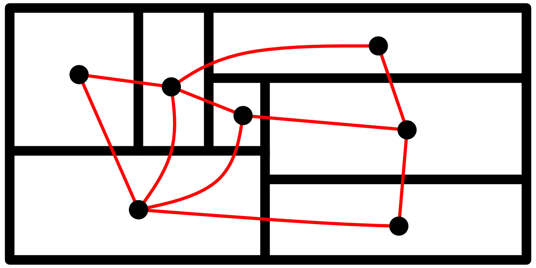
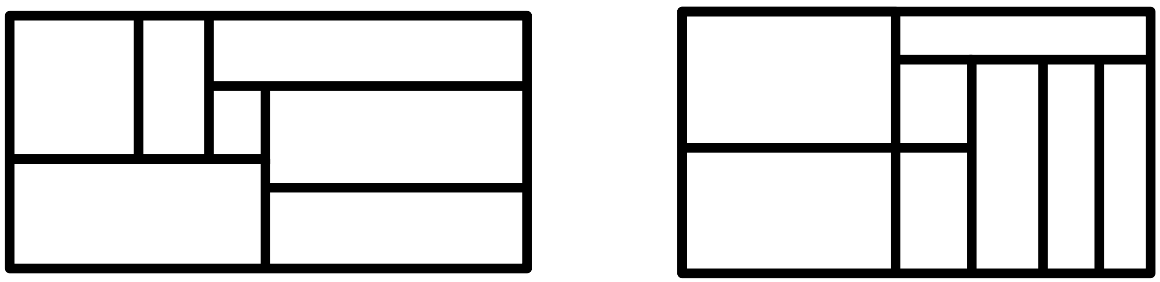


Rectangular Dual

Rectangular Subdivision System

Let R be a rectangle. A rectangular subdivision system Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



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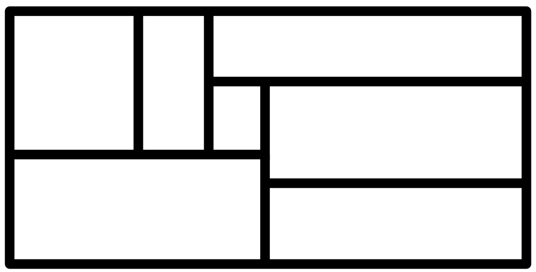

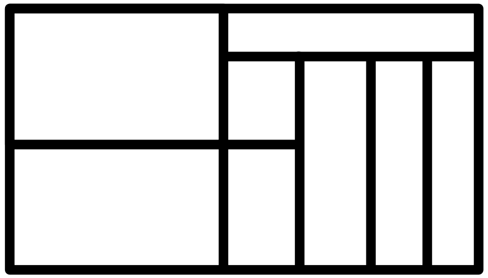

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A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \to \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to u and v, share a common boundary.

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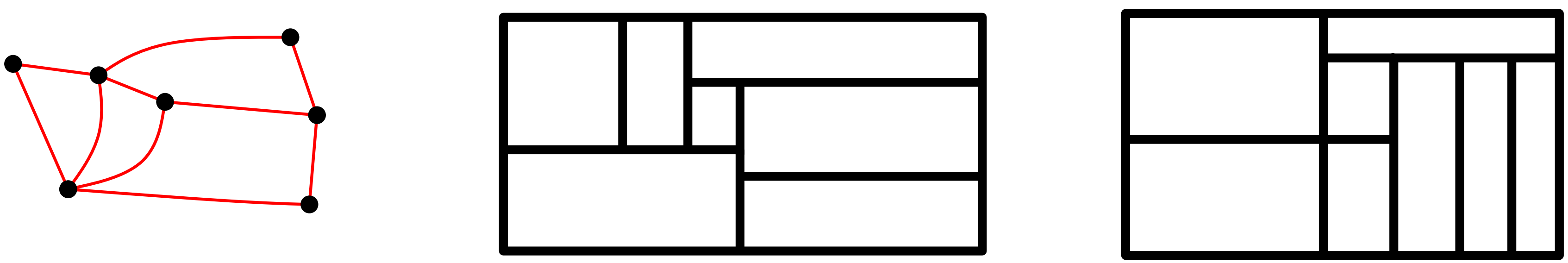

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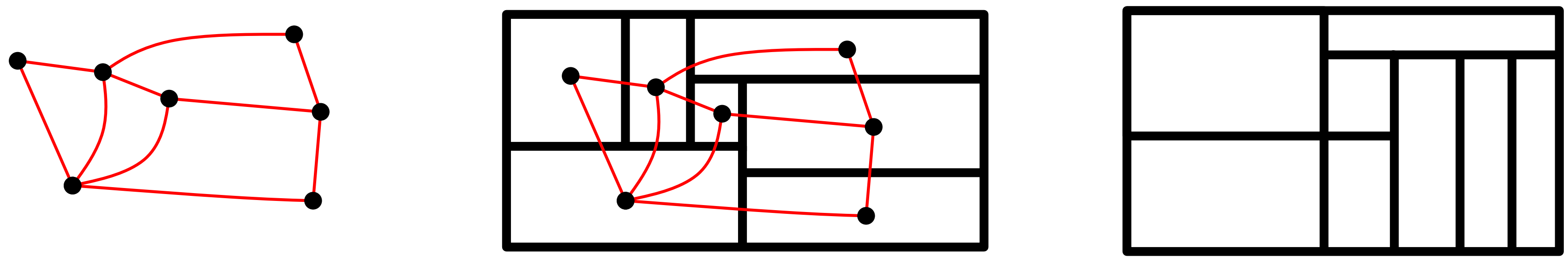

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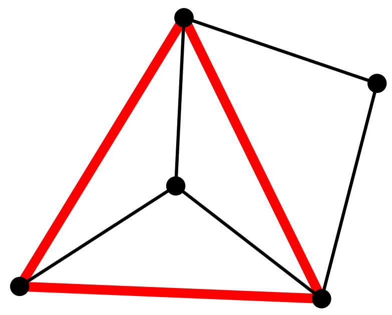
Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.



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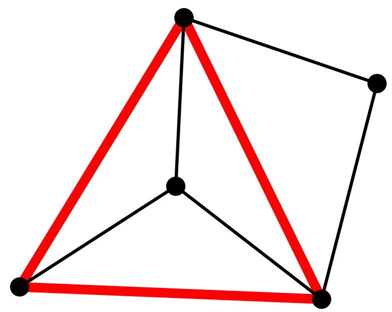

Does not have a rectangular dual!

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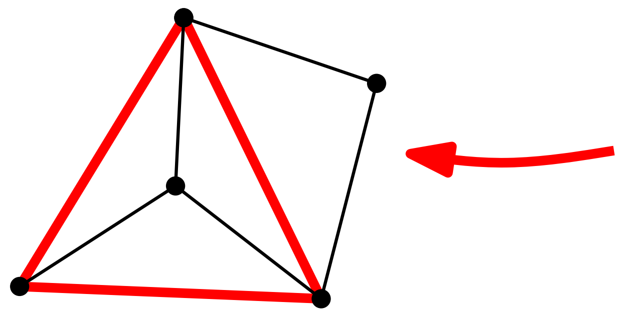

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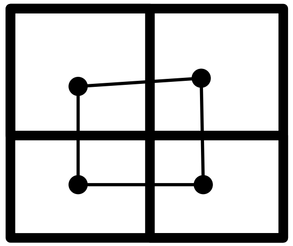

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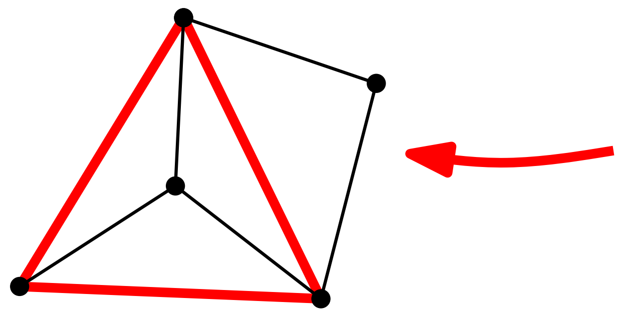

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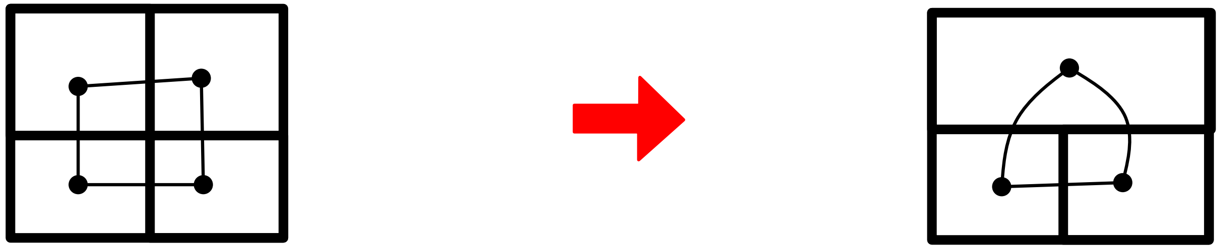
Each face of G must be a triangle!



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Necessary conditions for a planar graph G to have a rectangular dual:

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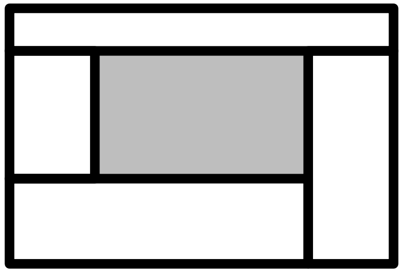
Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

- Every interior face of G is a triangle and the exterior face of G is a quadrangle;
- G has no separating triangles



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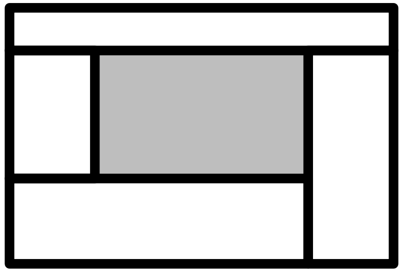
Proper Triangular Planar Graph (PTP)



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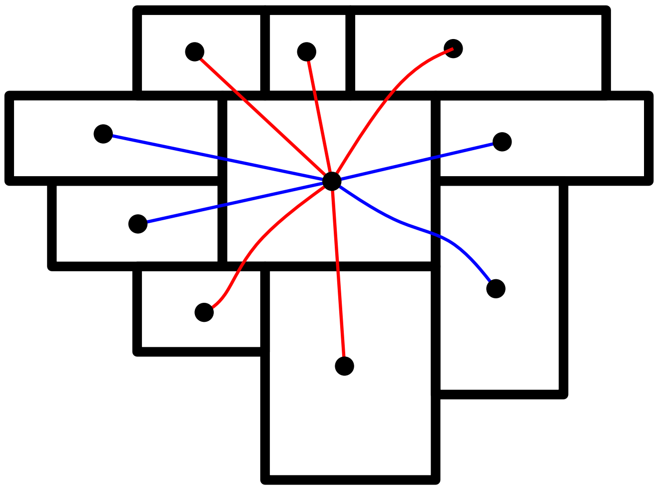

In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.

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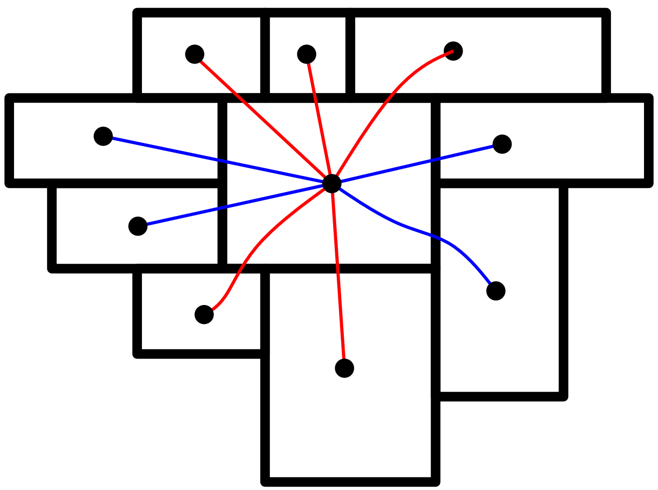

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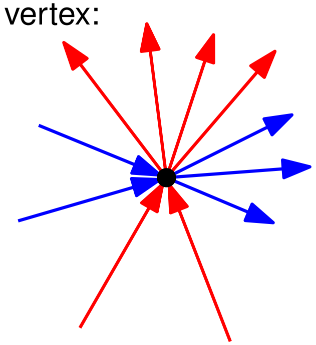
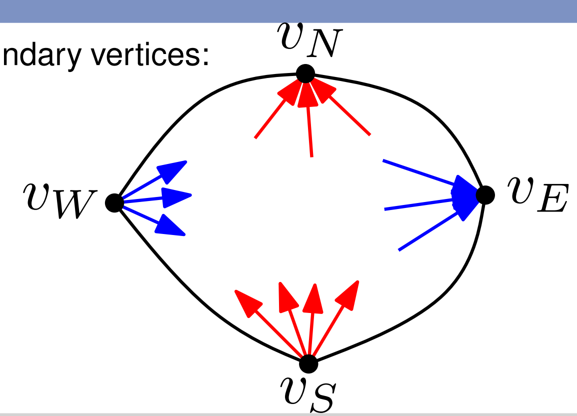
Regular edge labeling

For each internal vertex:



For the boundary vertices:



Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
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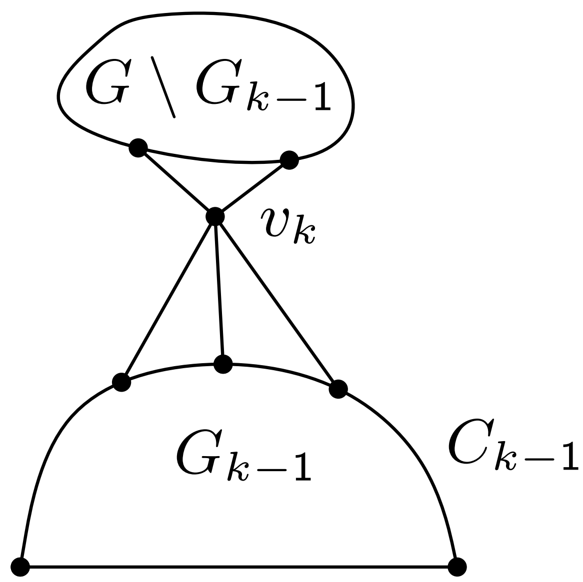

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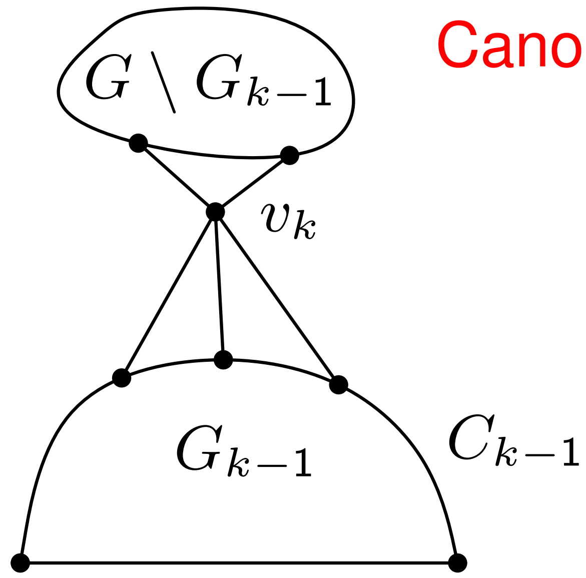

Canonical ordering with extra condition on v_k!

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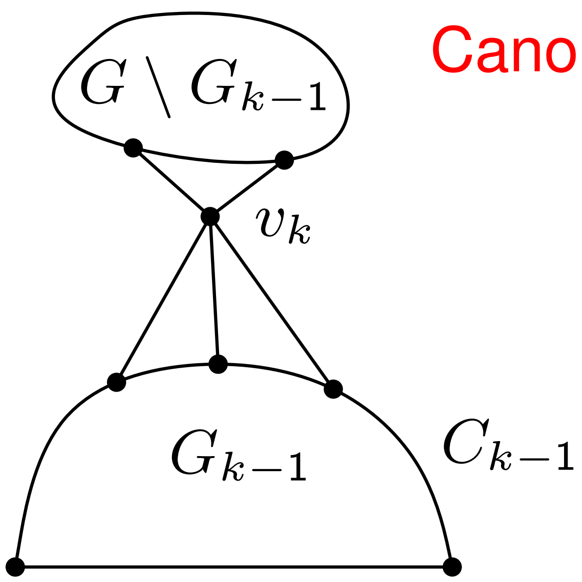

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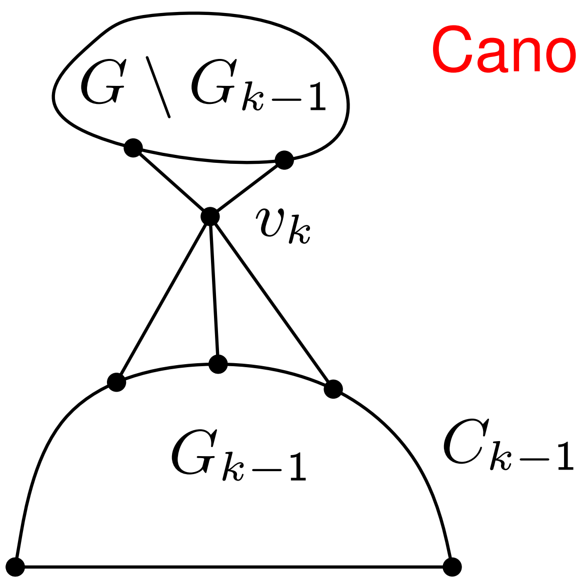
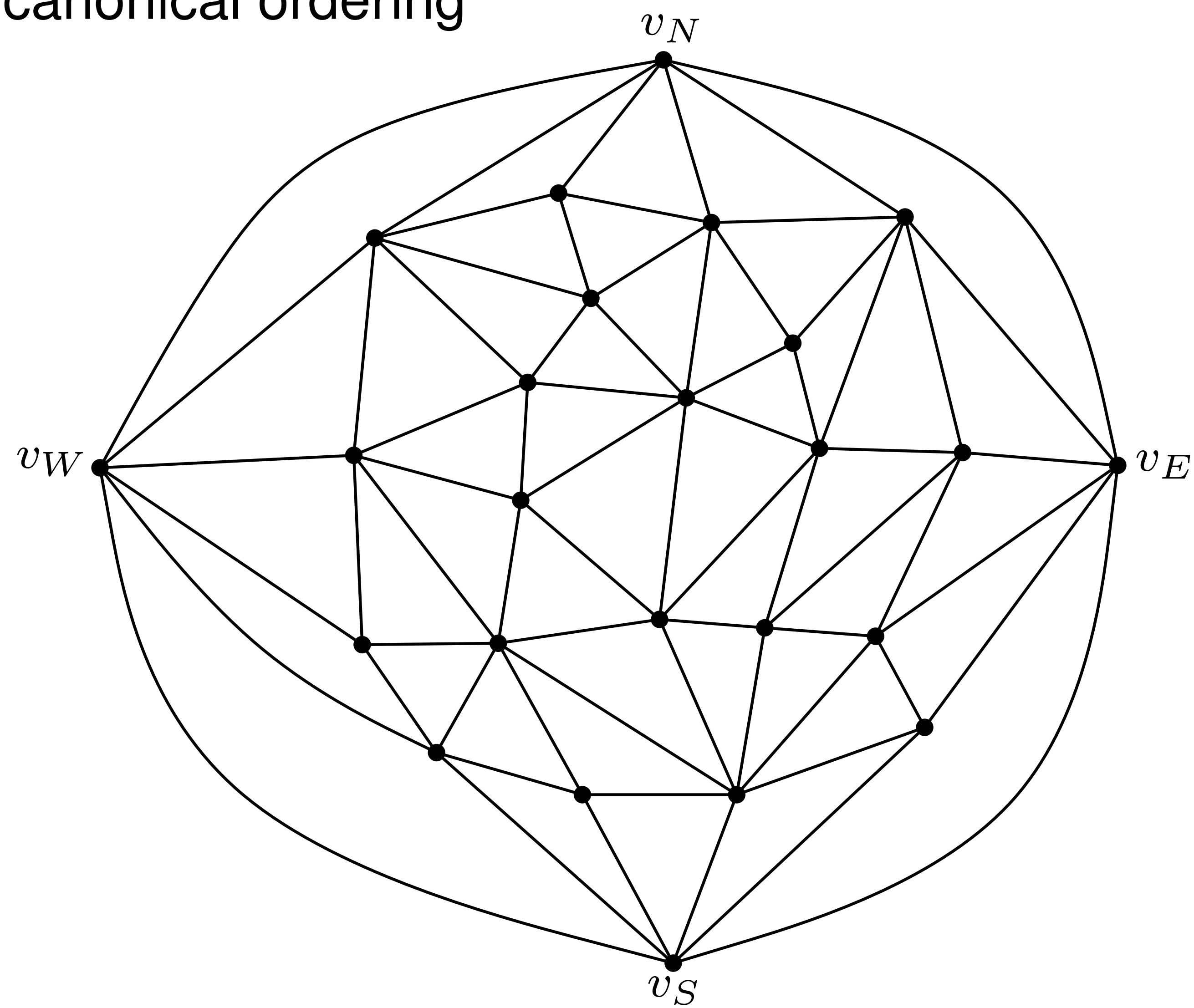


Canonical ordering with extra condition on v_k!

Home task!

Rectangular Dual

Refined canonical ordering



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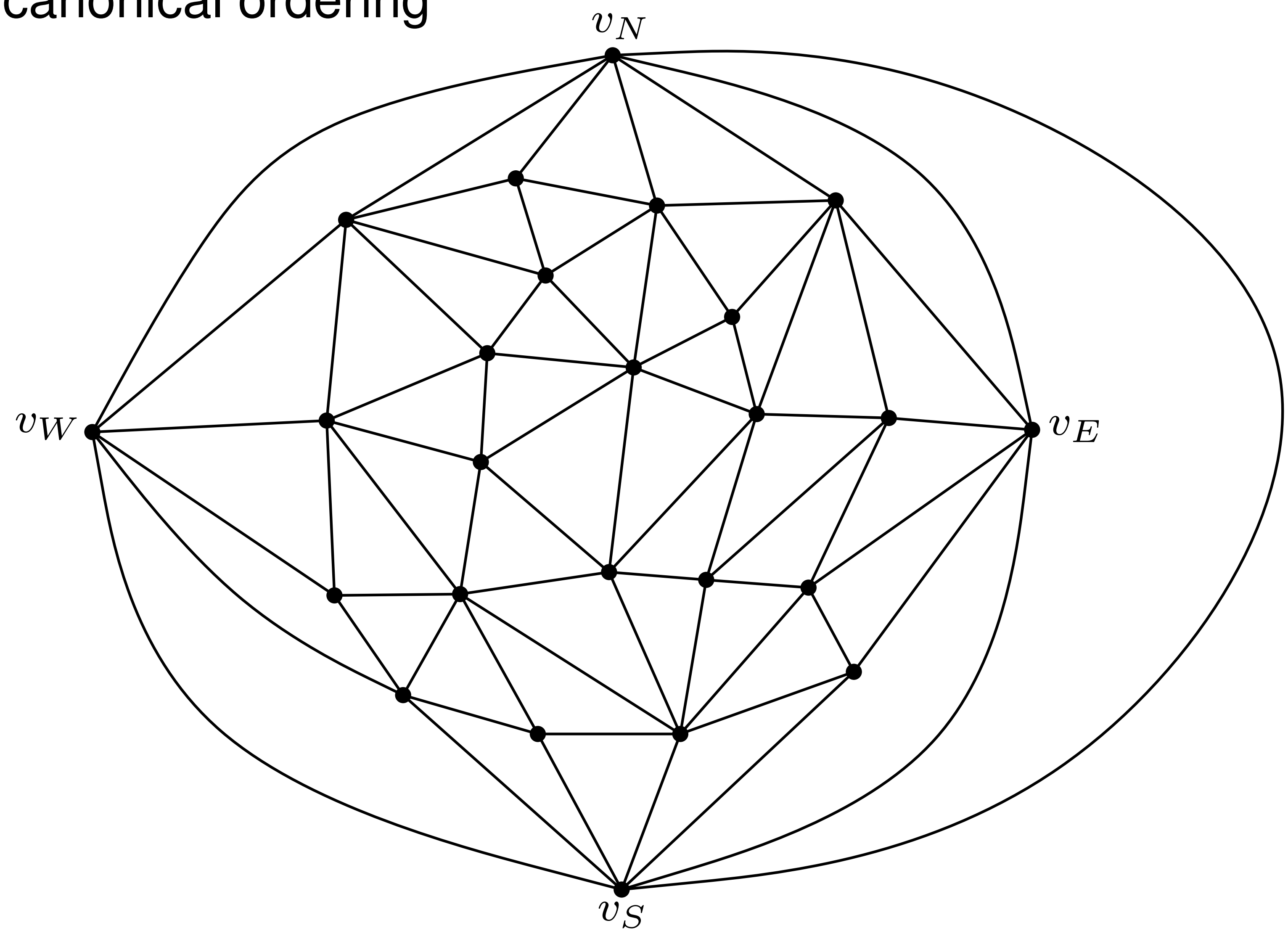

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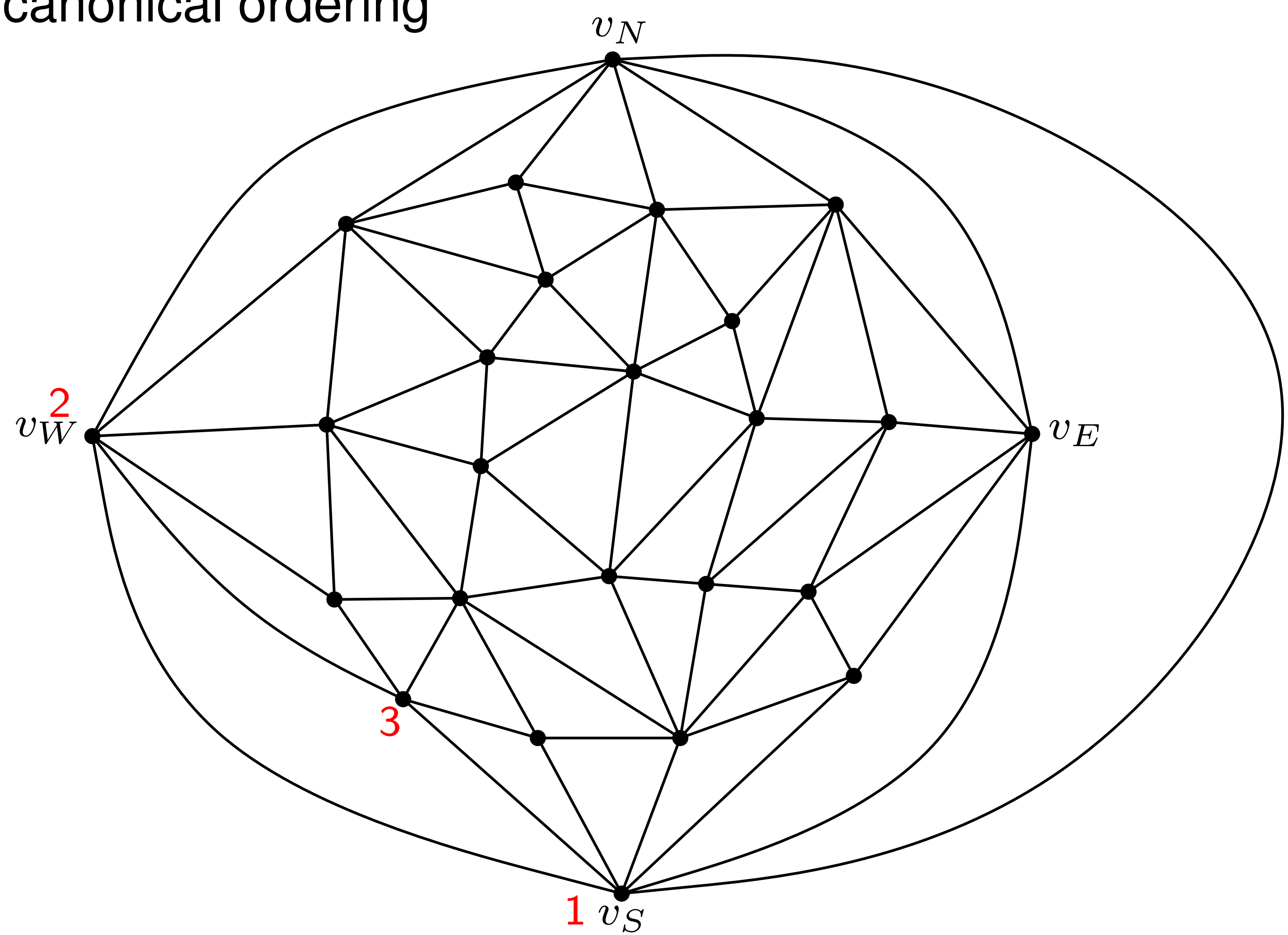

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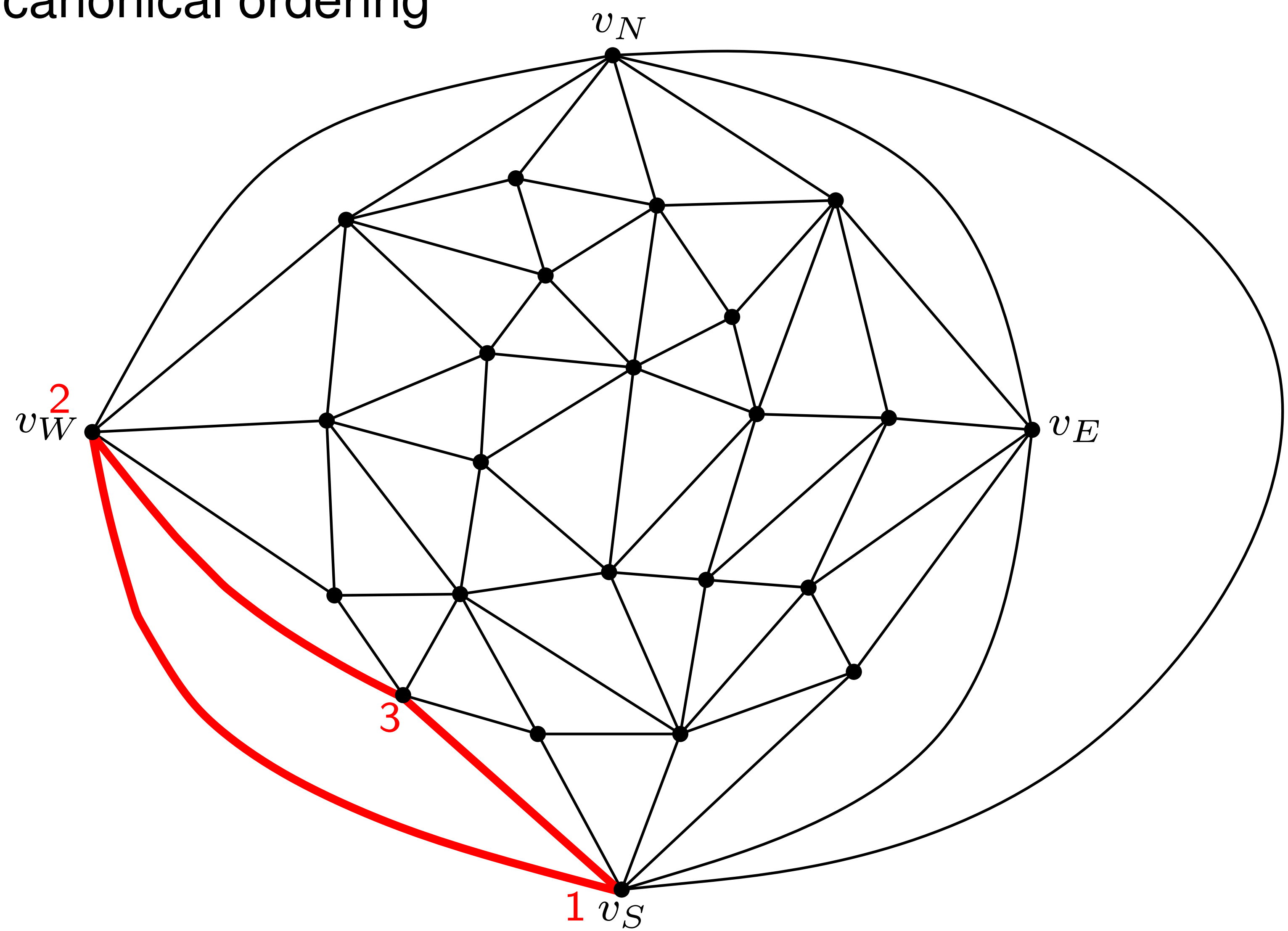

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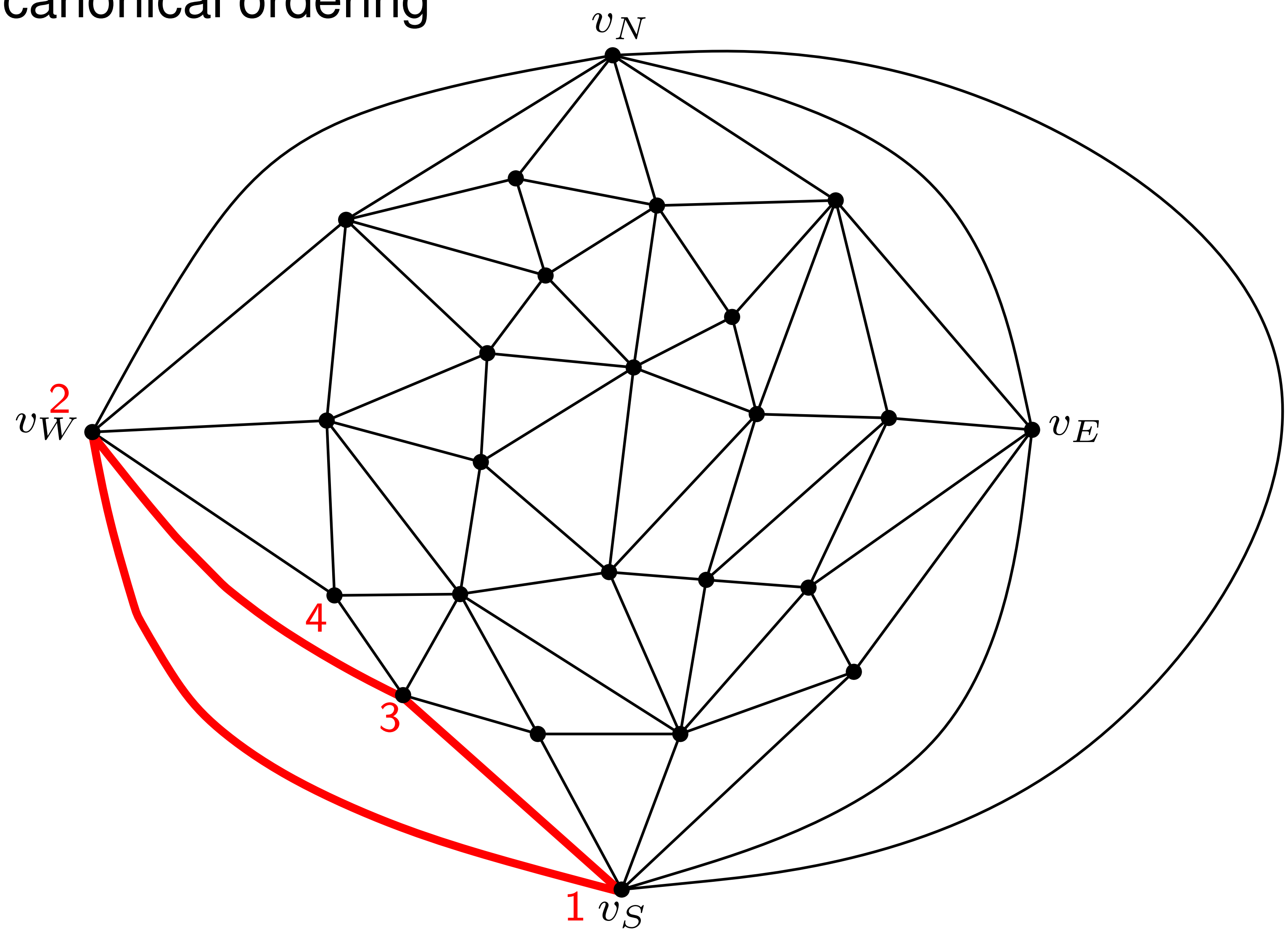

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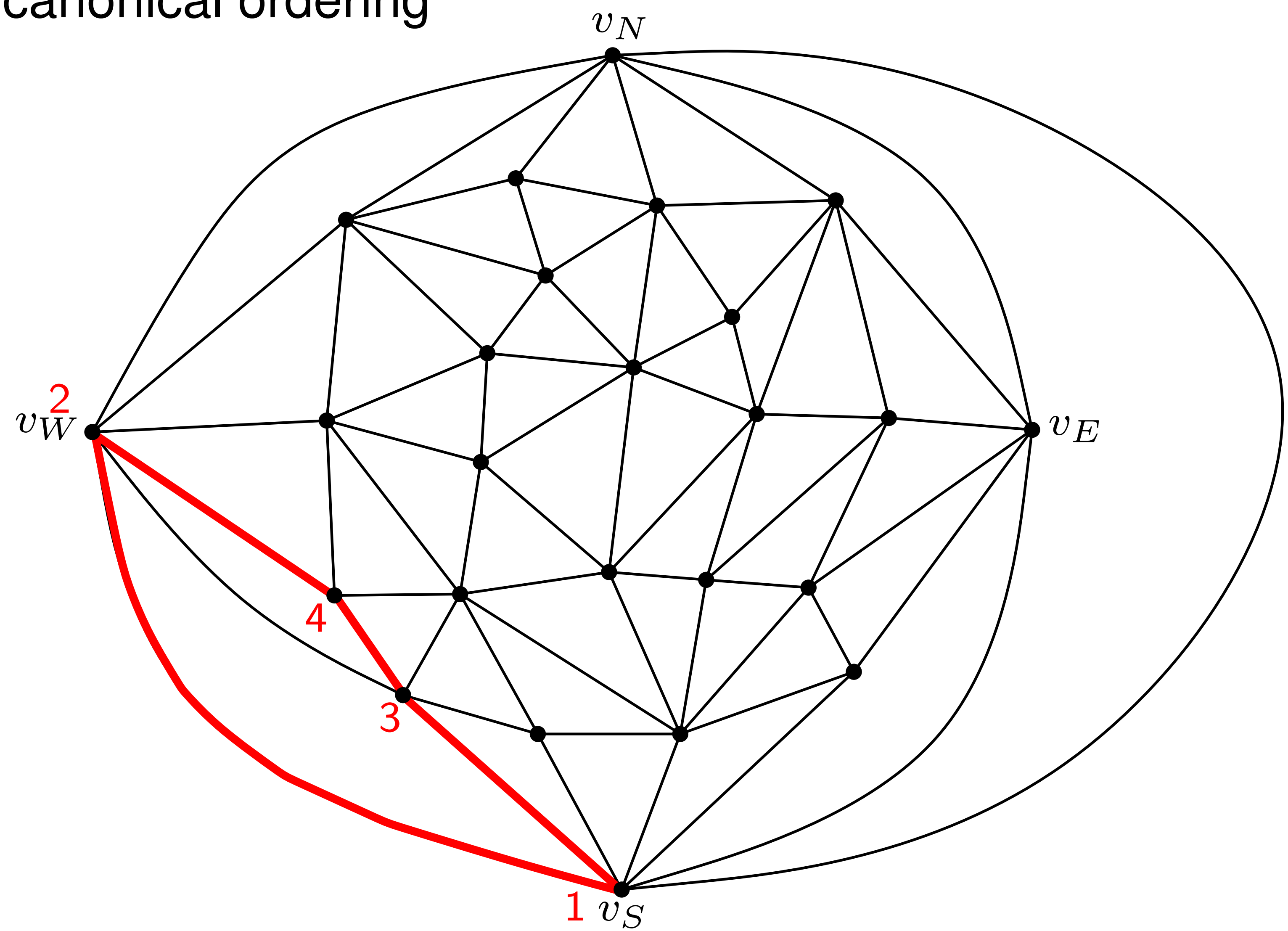

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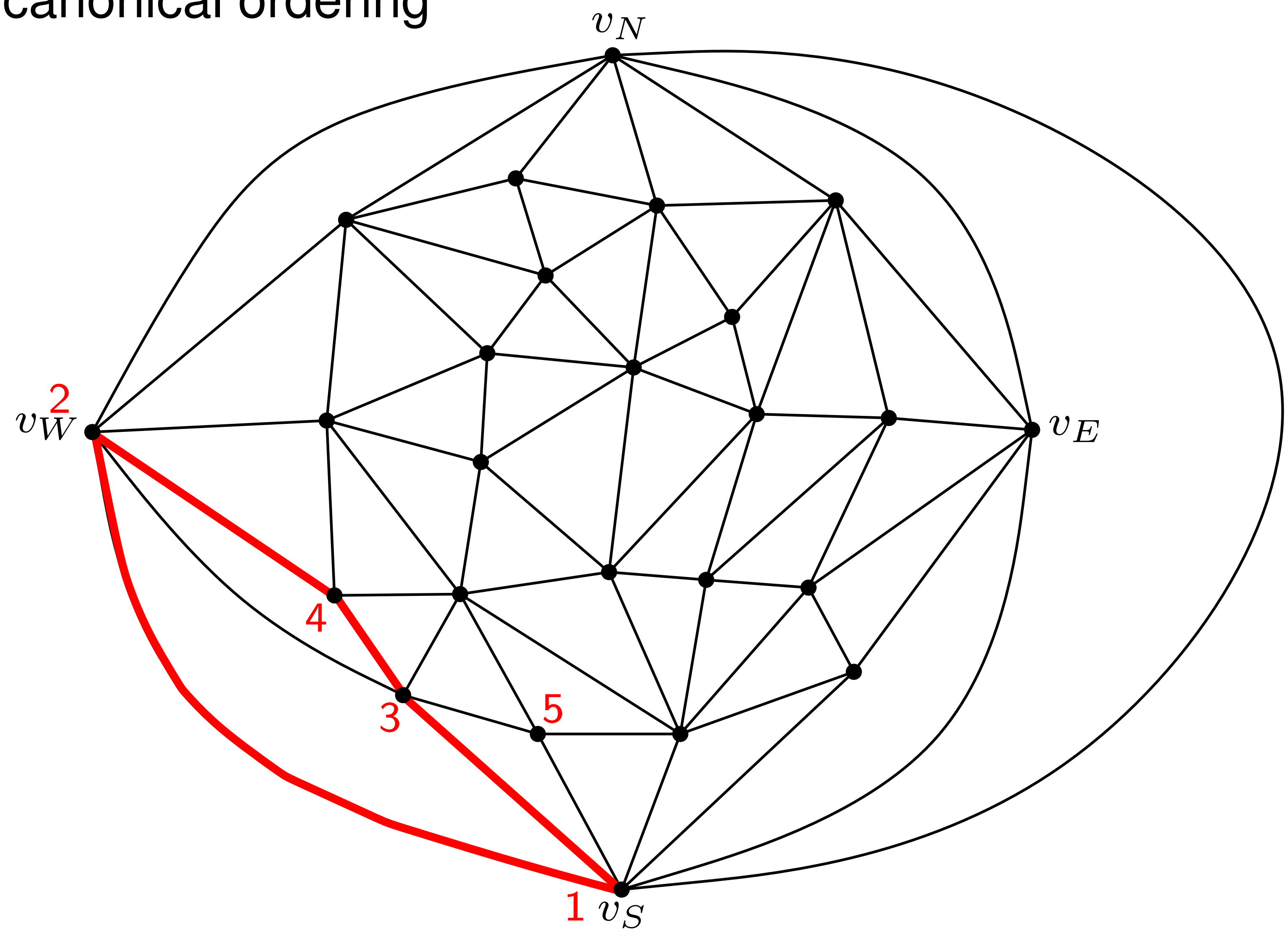

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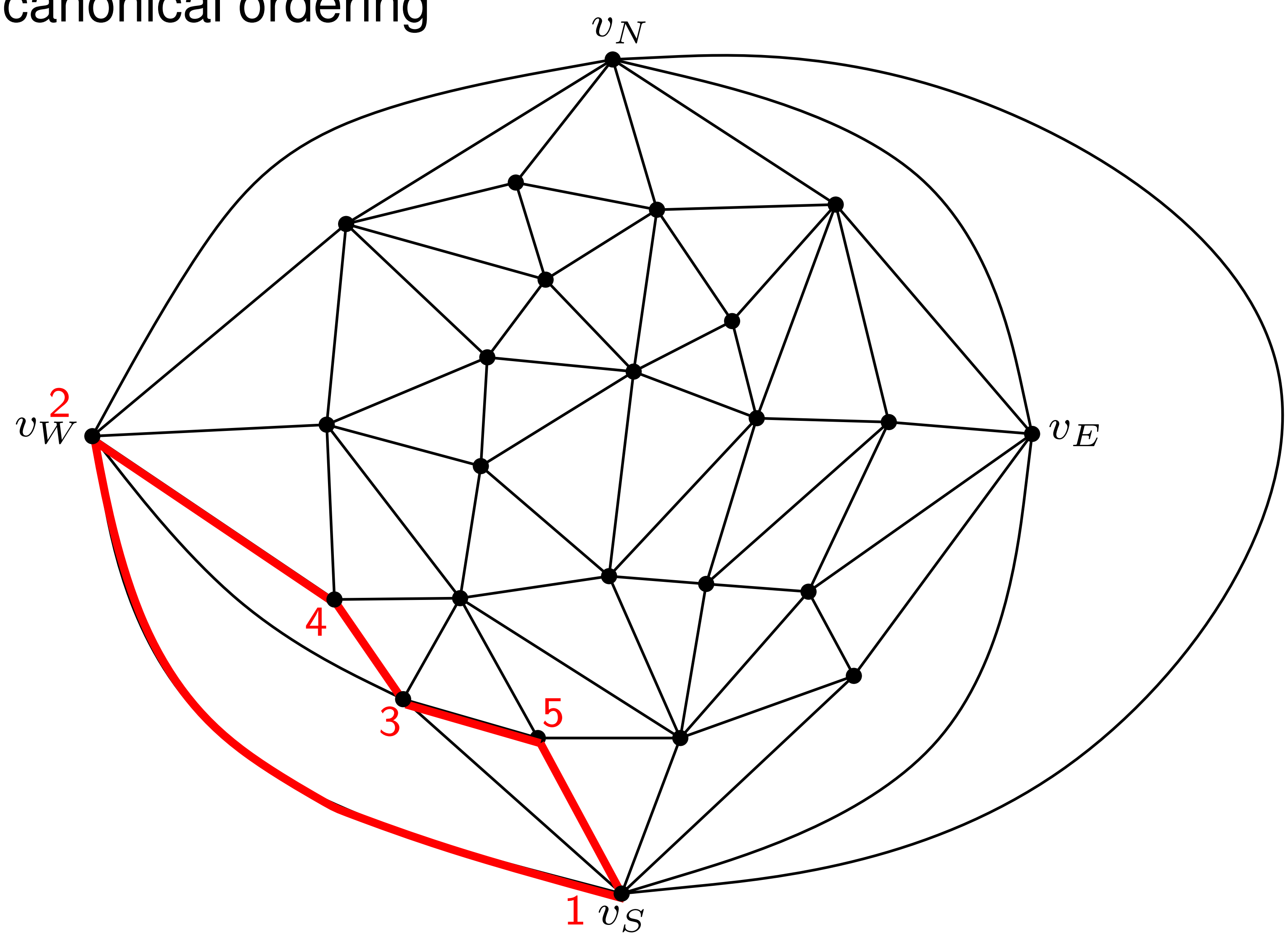

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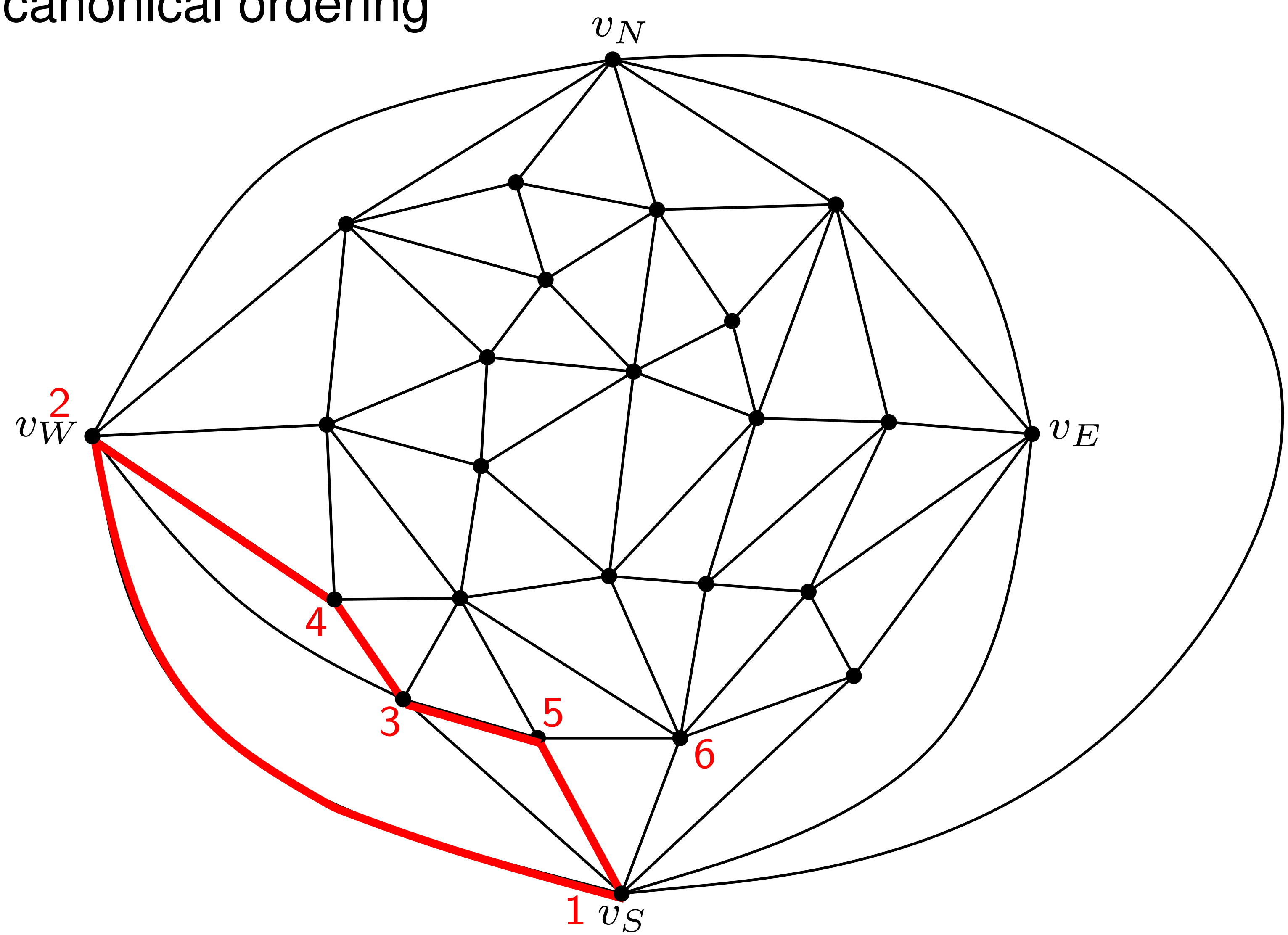

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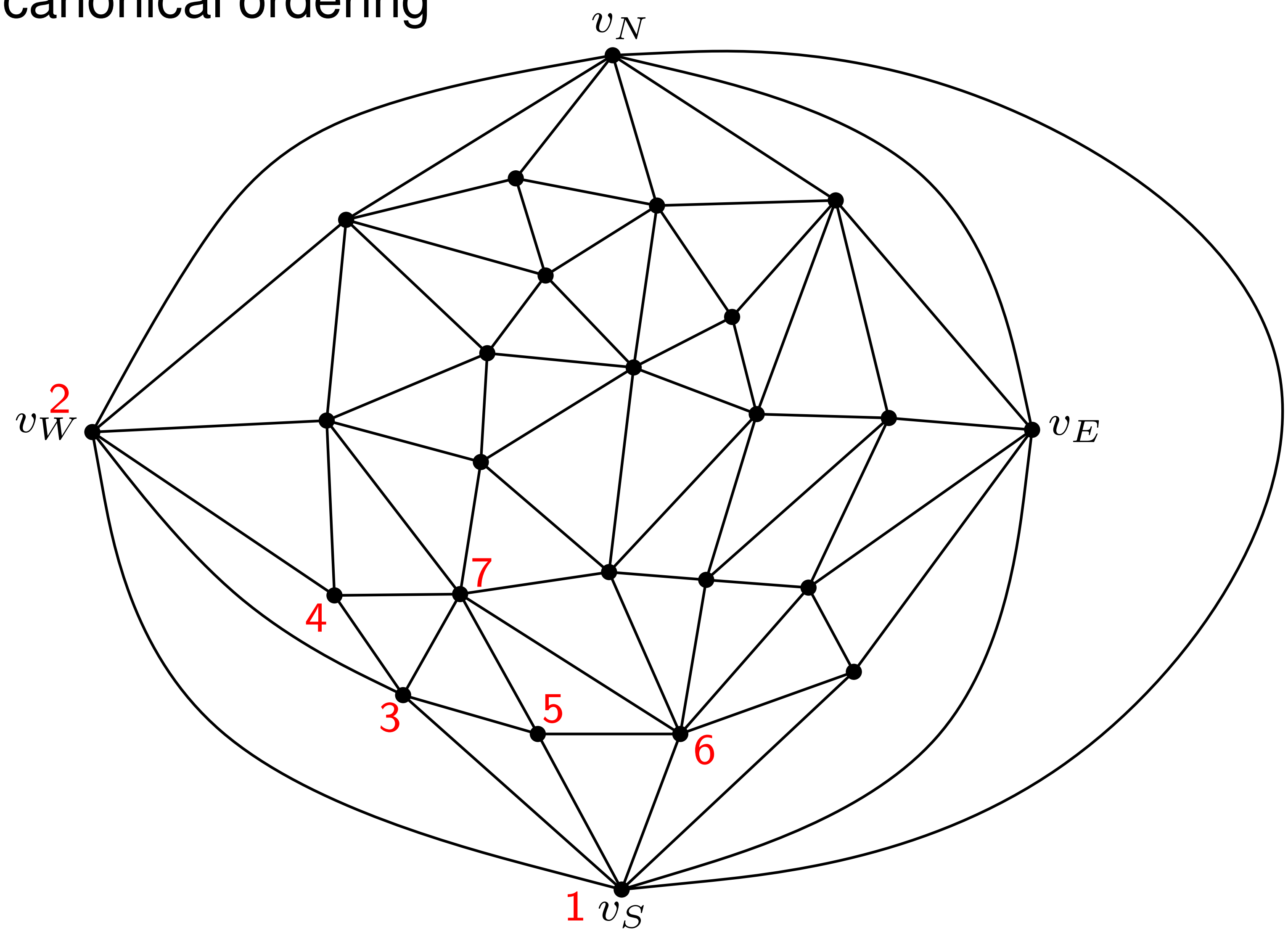

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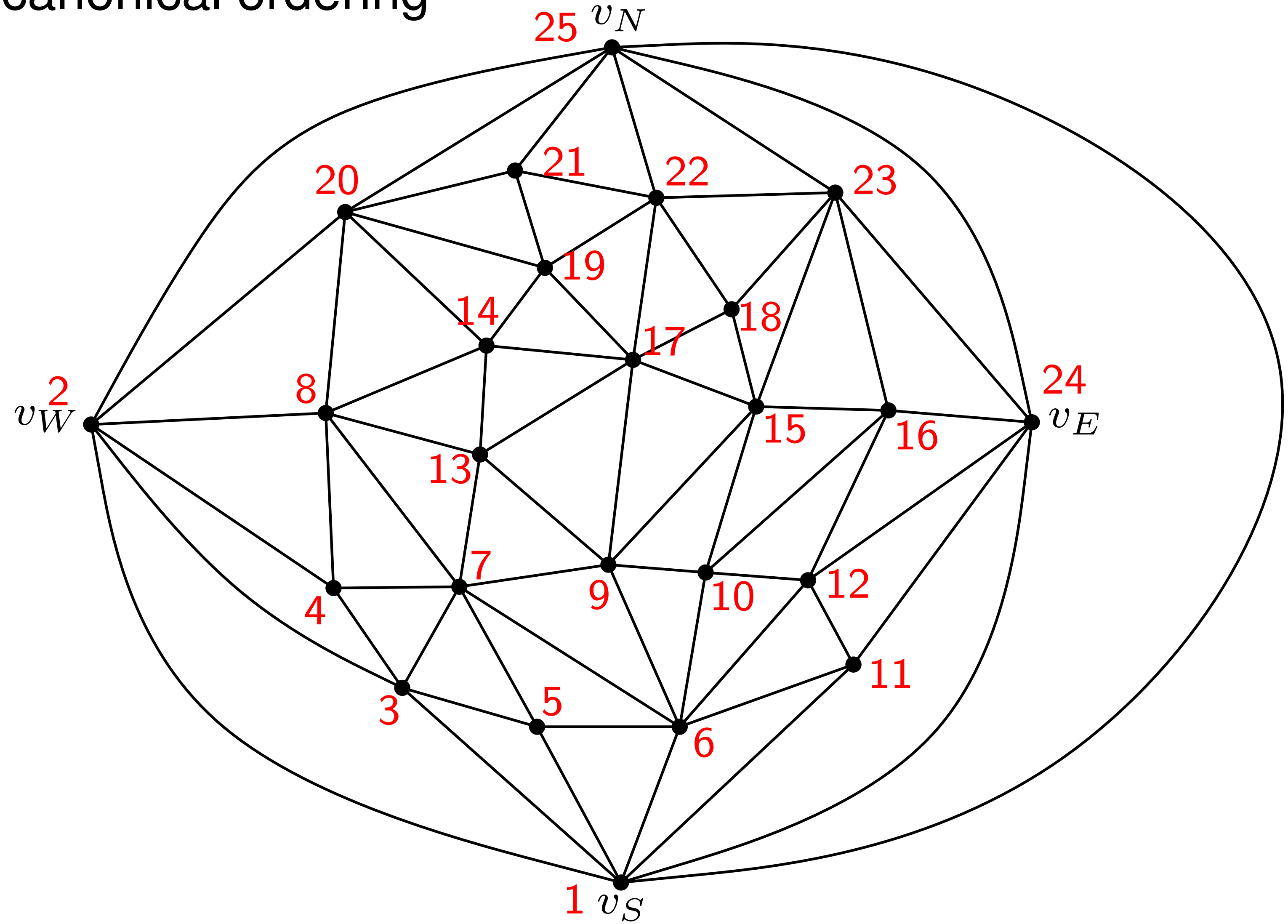

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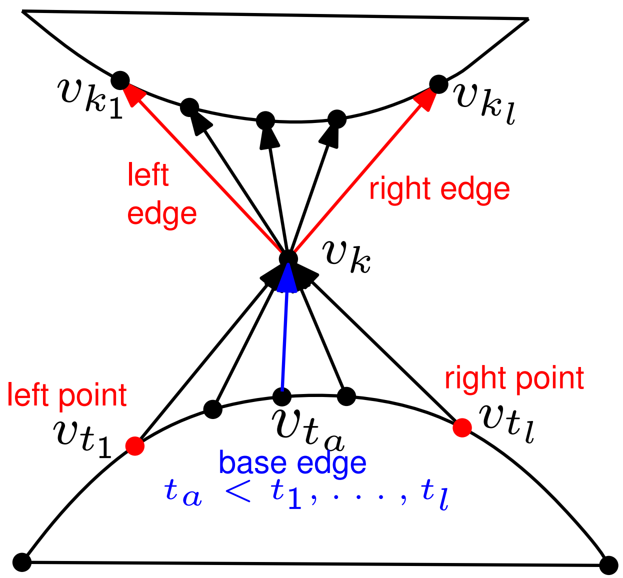
Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



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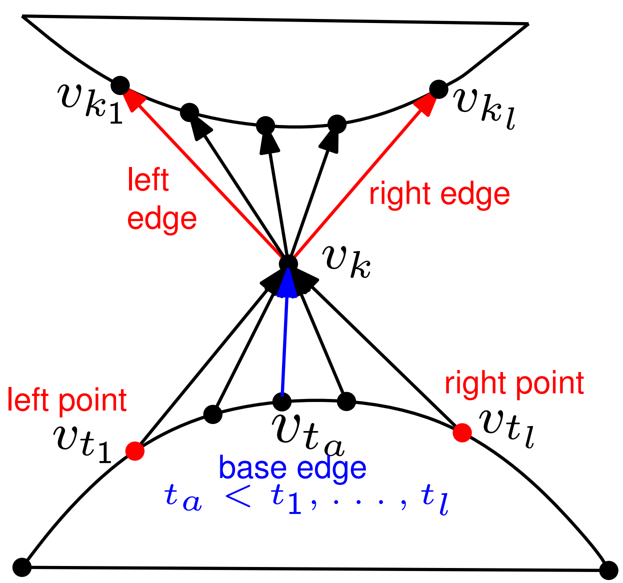

Lemma 1

A base edge can not be a left edge or right edge.

Rectangular Dual

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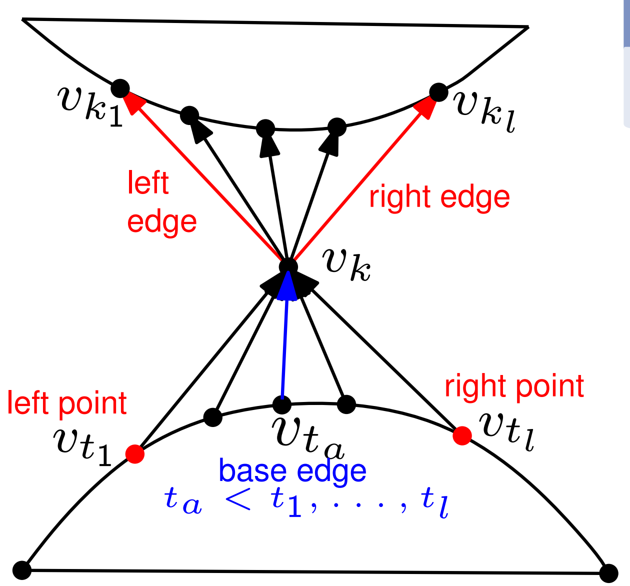

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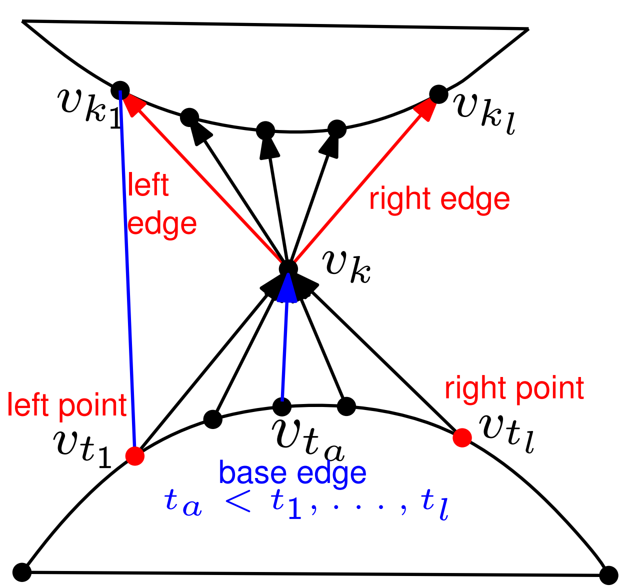

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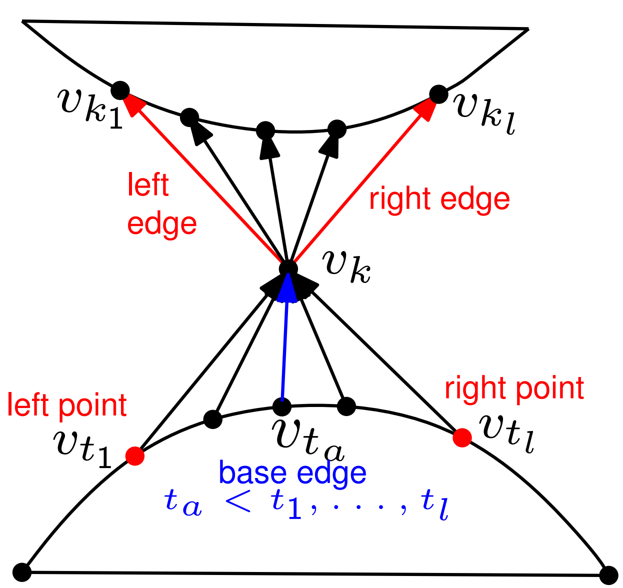

Lemma 2

An edge is either a left edge, a right edge or a base edge.

Proof:

- The exclusive "or" follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i}, $1 \leq i < a-1$
- Edges (v_{t_i}, v_k), $1 \leq i < a-1$, are right edges;
- Similarly we prove that edges (v_{t_i}, v_k), $a+1 \leq i < l$, are left edges;

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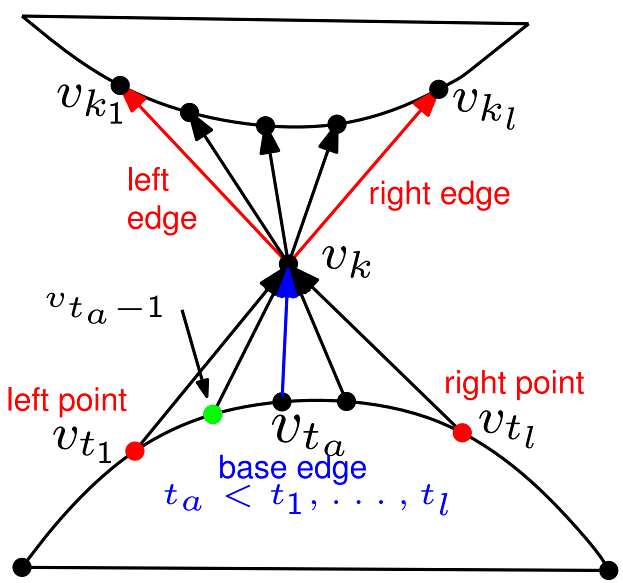

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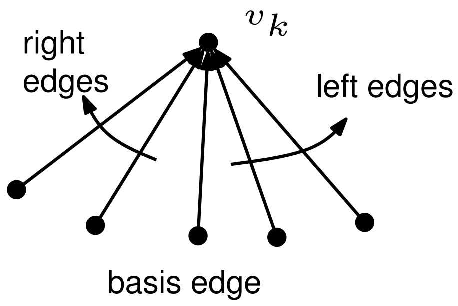

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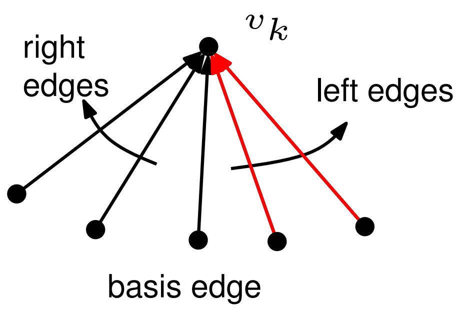

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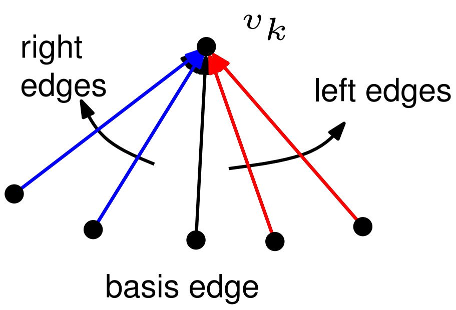

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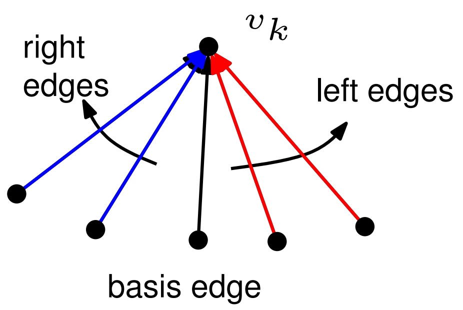

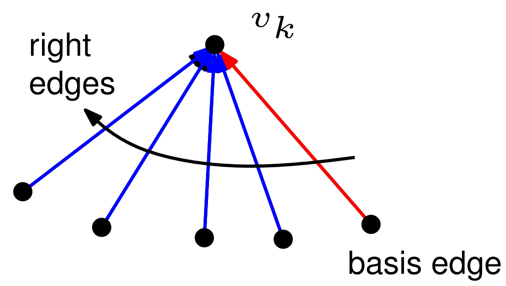

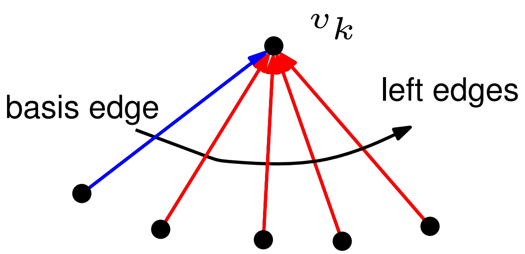

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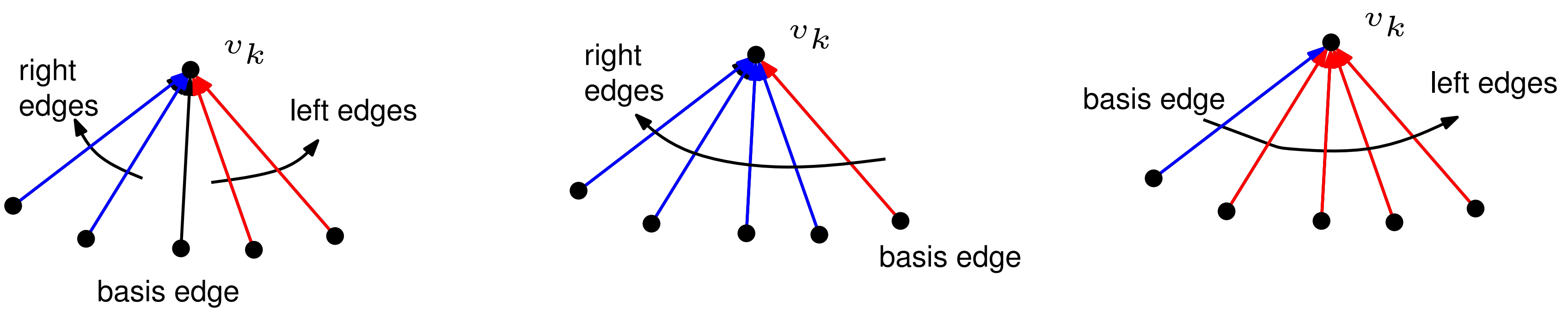

We call T_b blue edges and T_r red edges.

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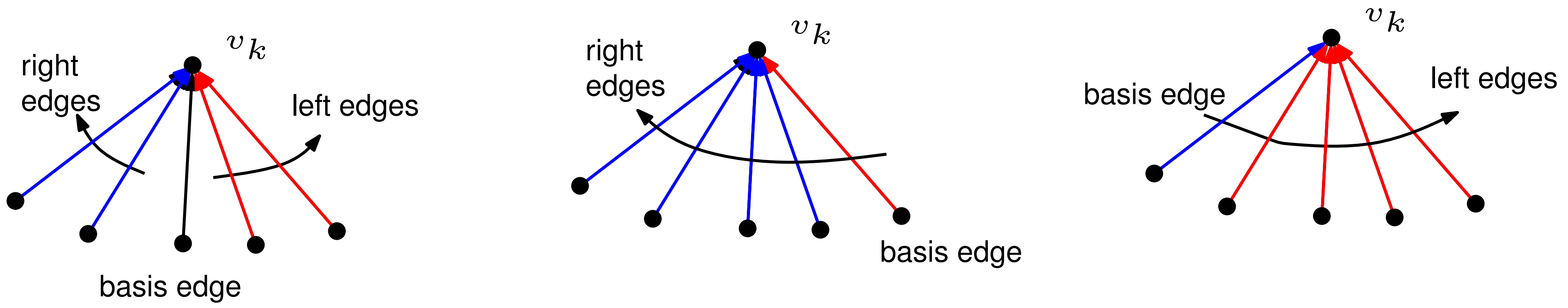

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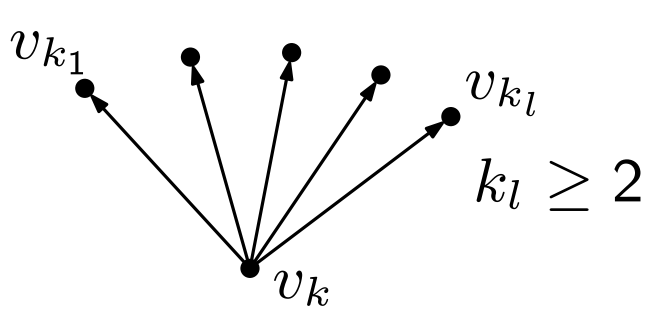

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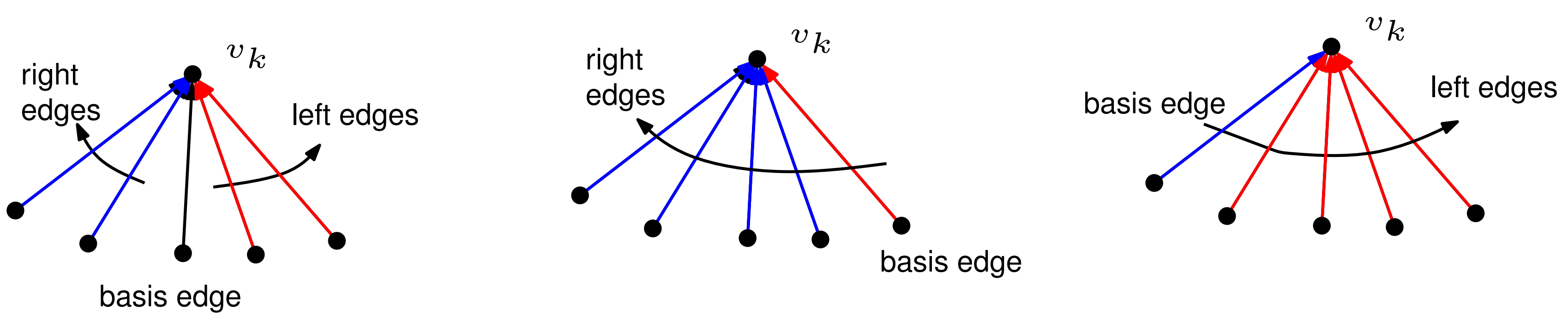

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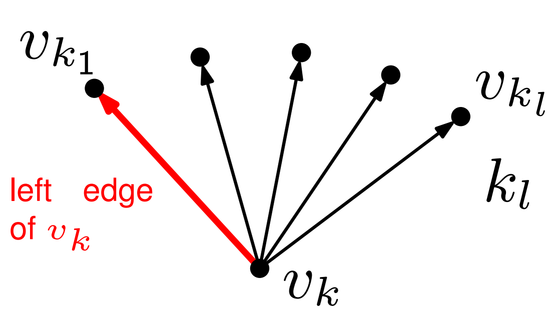
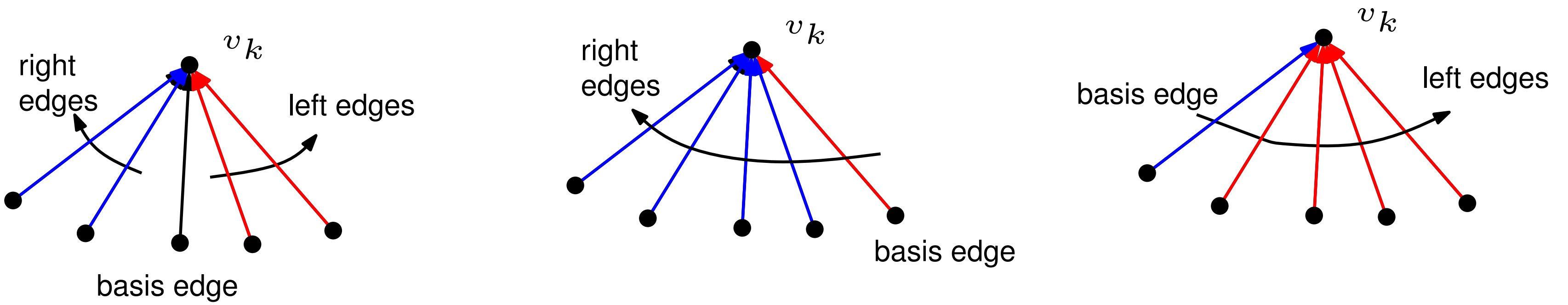


$k_l \geq 2$

Rectangular Dual



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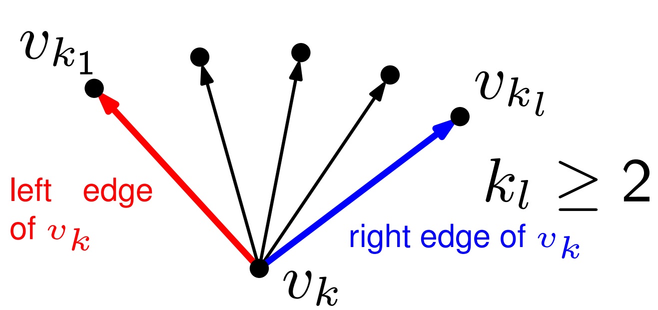

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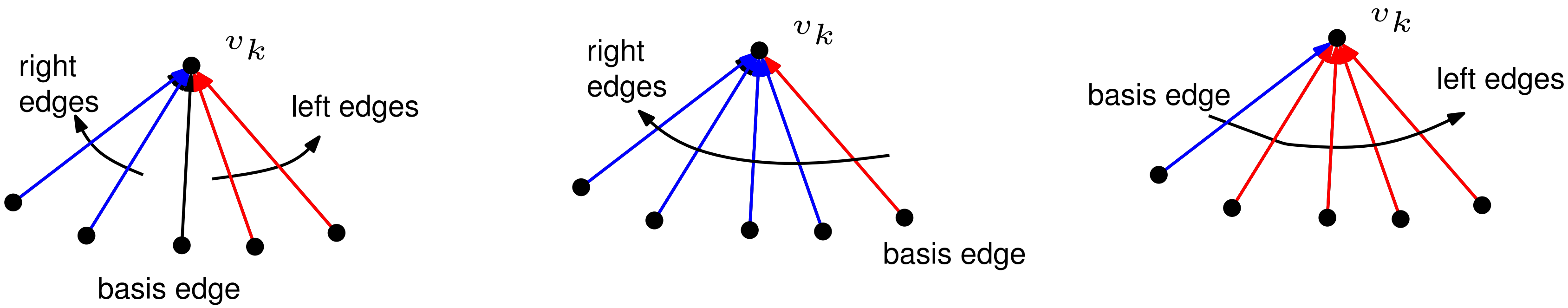

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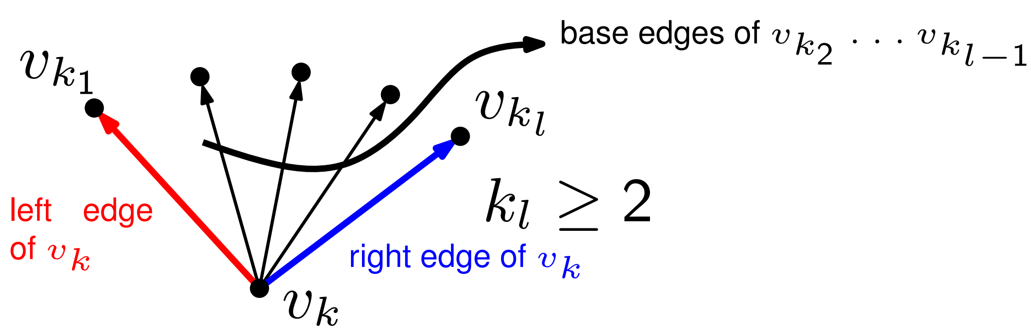

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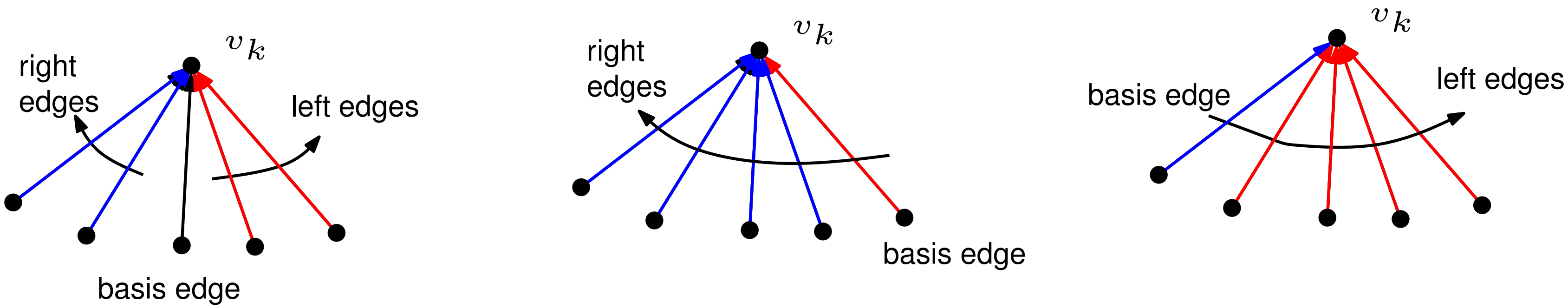

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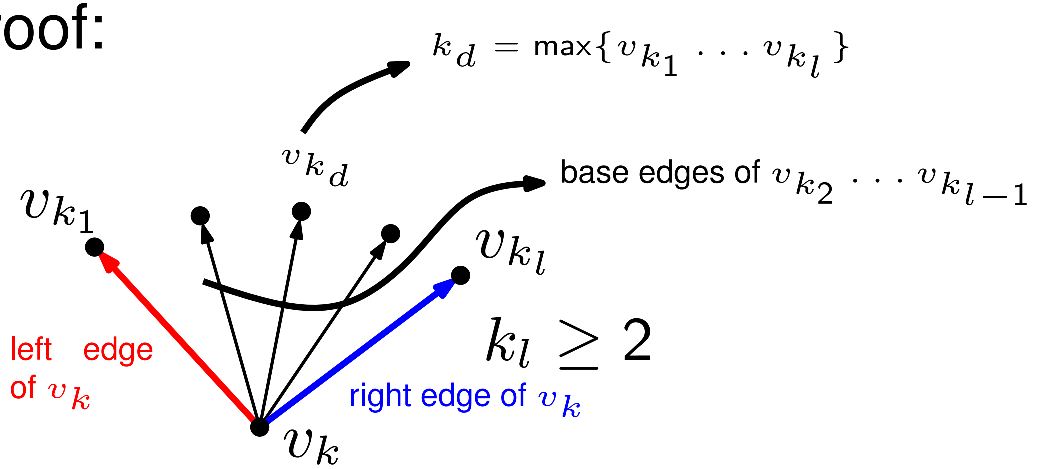

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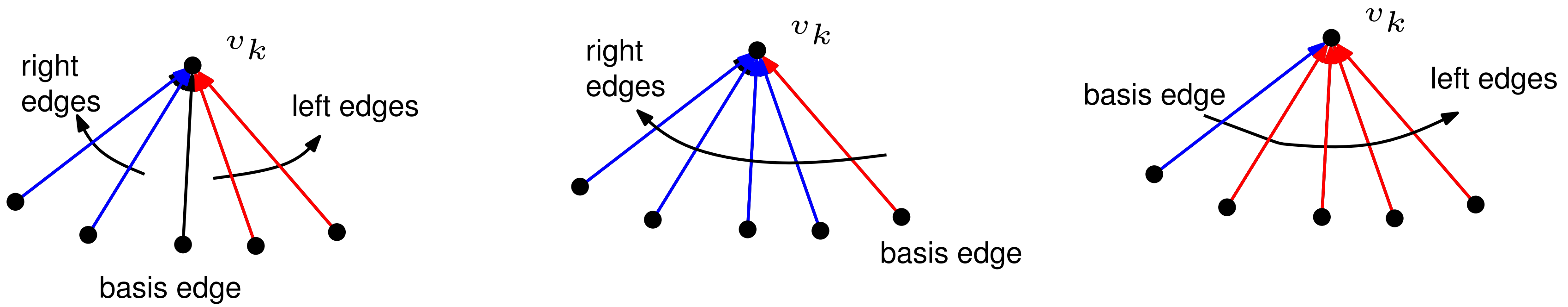

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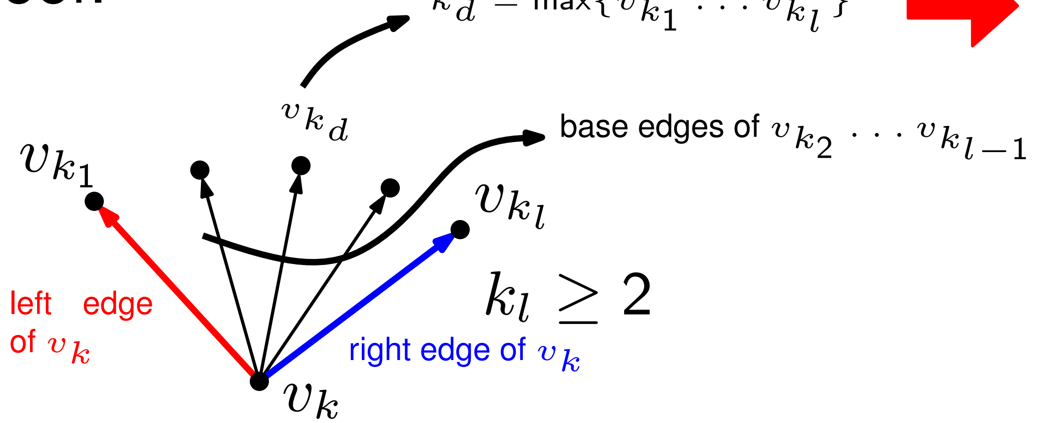
$k_d = \max\{v_{k_1} \dots v_{k_l}\}$ ⟹ $k_1 < k_2 < \dots < k_d$ and $k_d > k_{d+1} > \dots > k_l$



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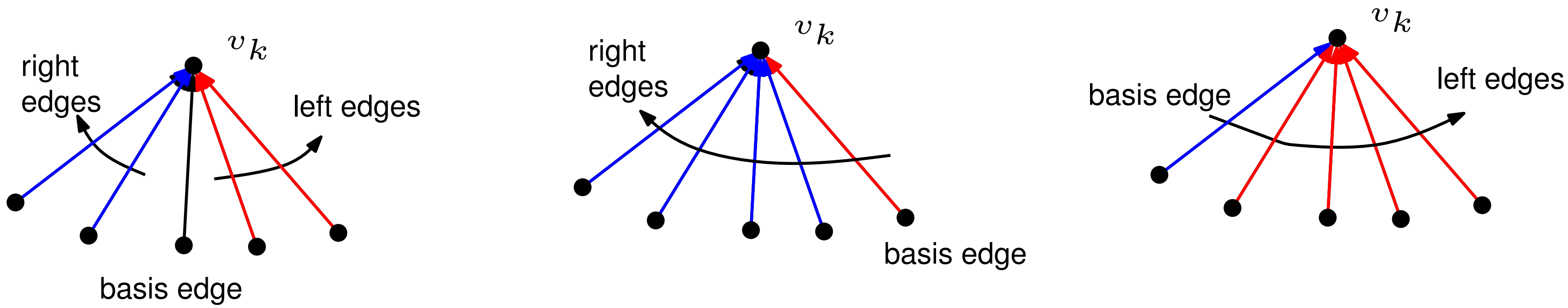

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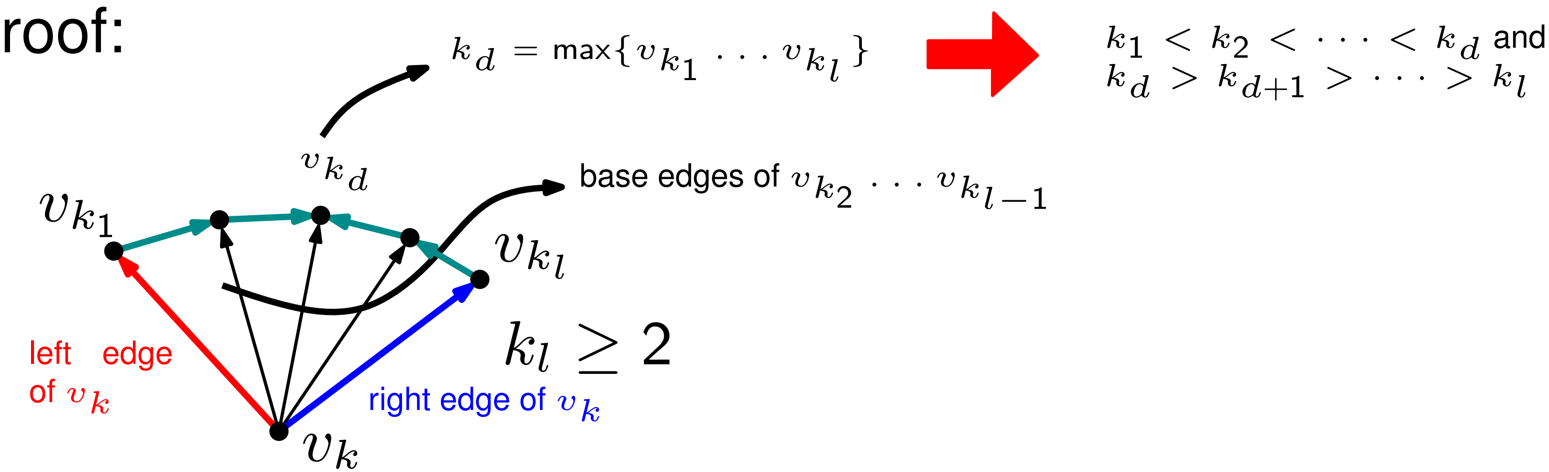

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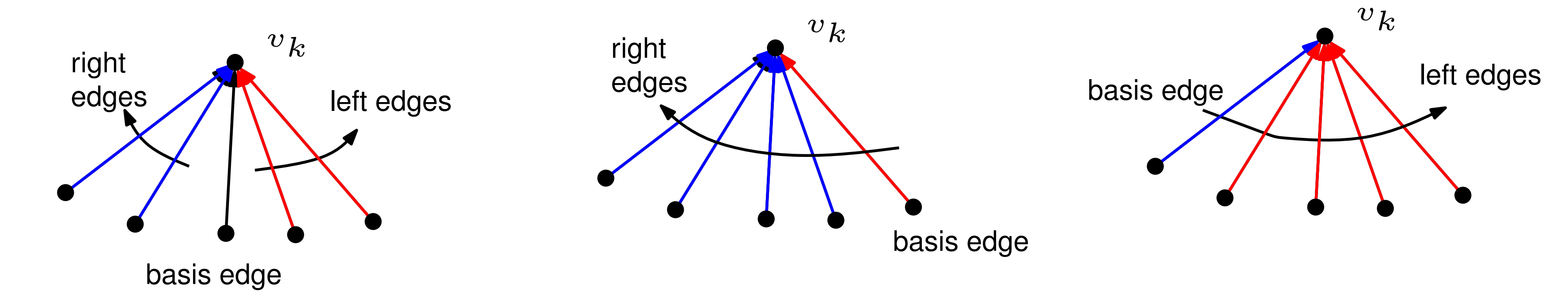

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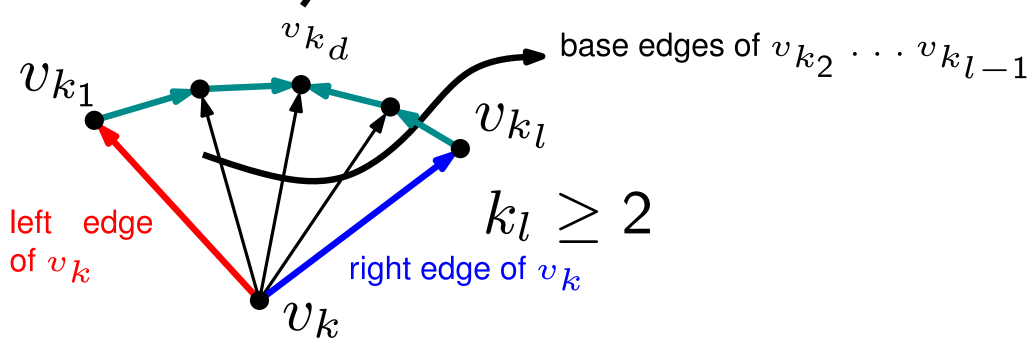

$k_1 < k_2 < \dots < k_d$ and
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(v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
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edge (v_k, v_{k_d}) is either red or blue

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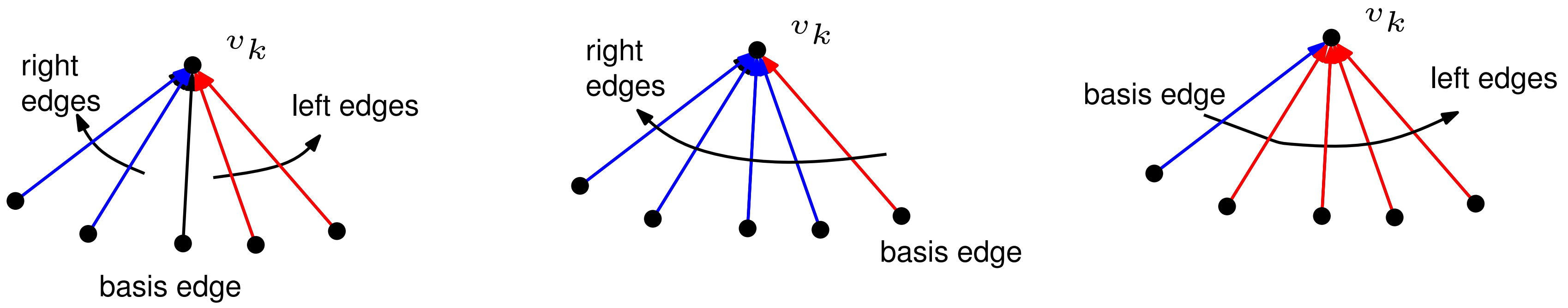

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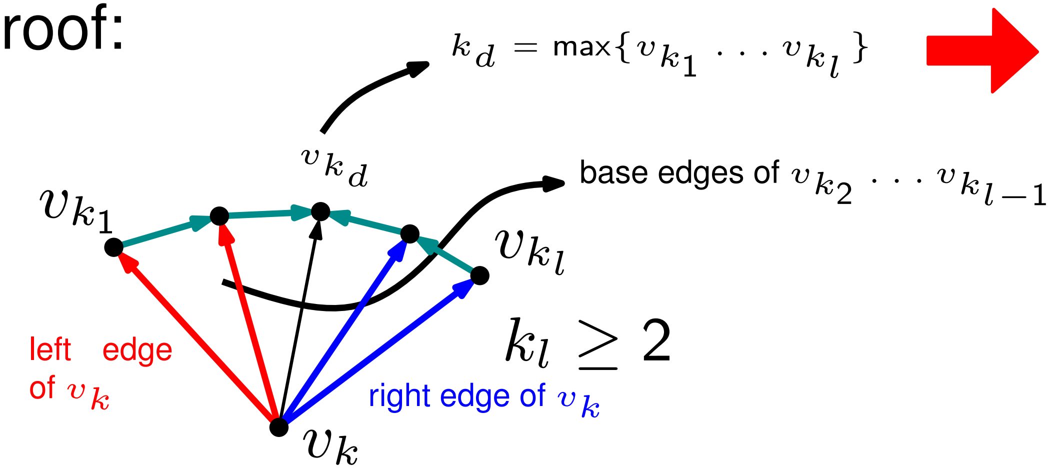

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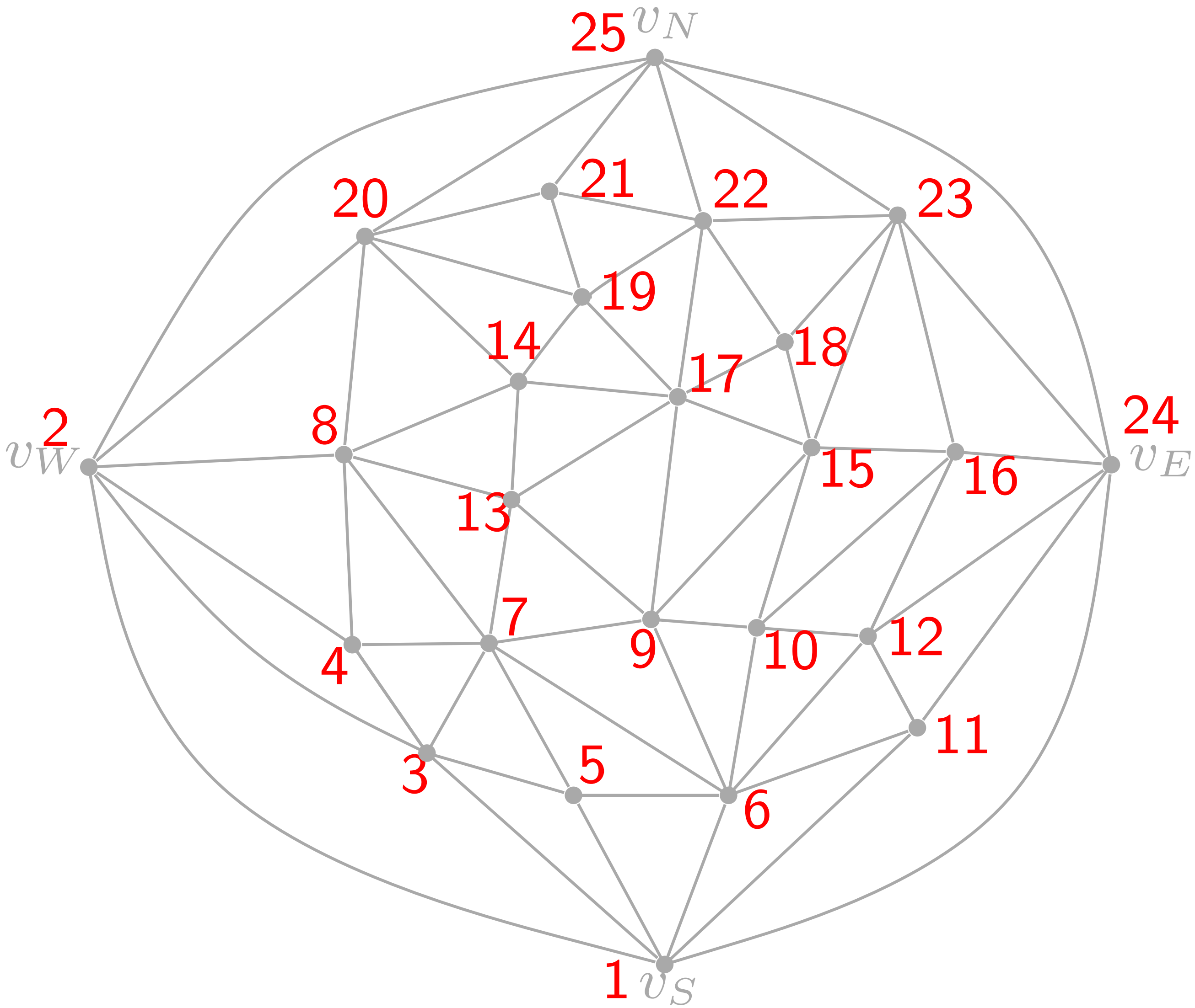

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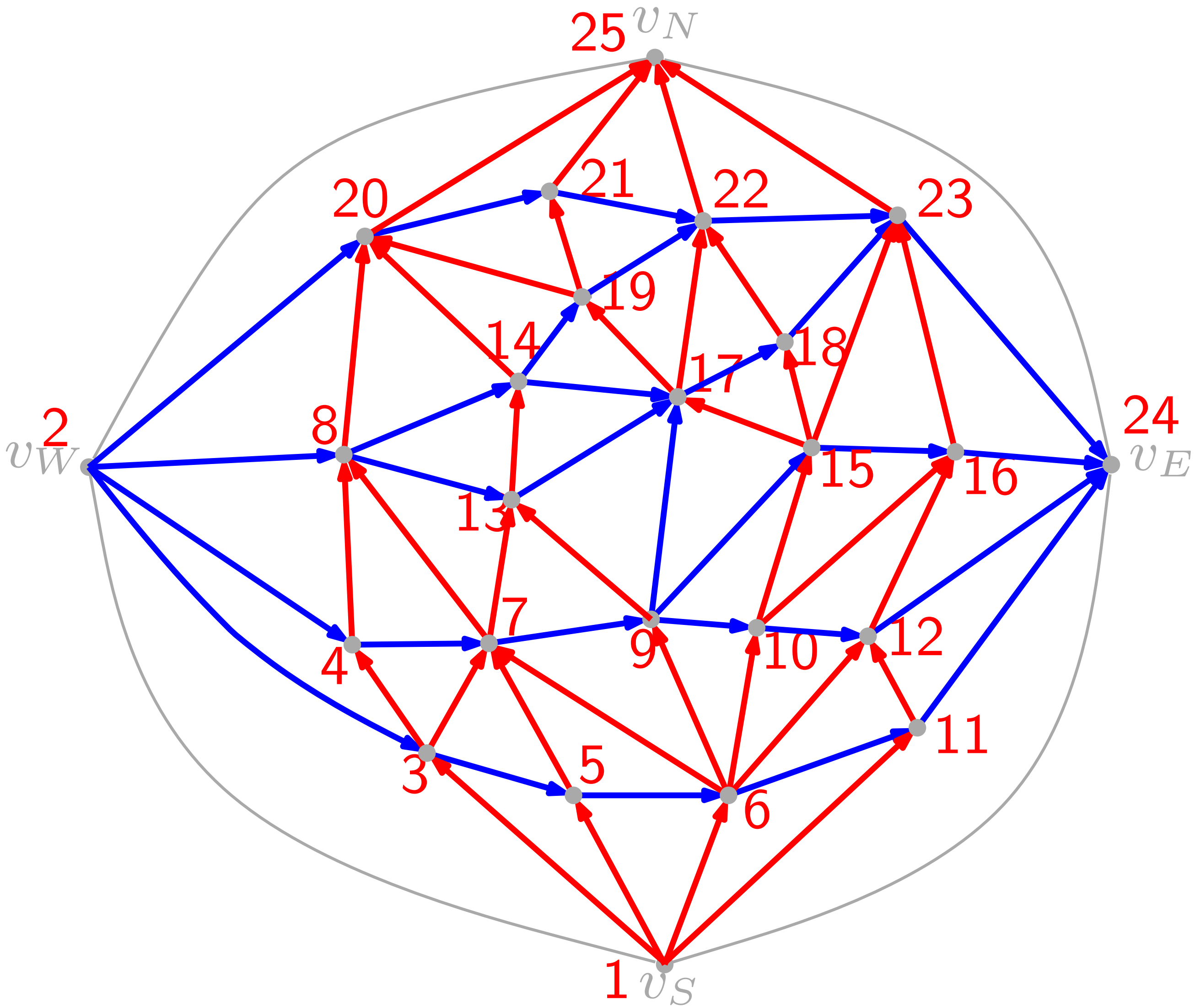
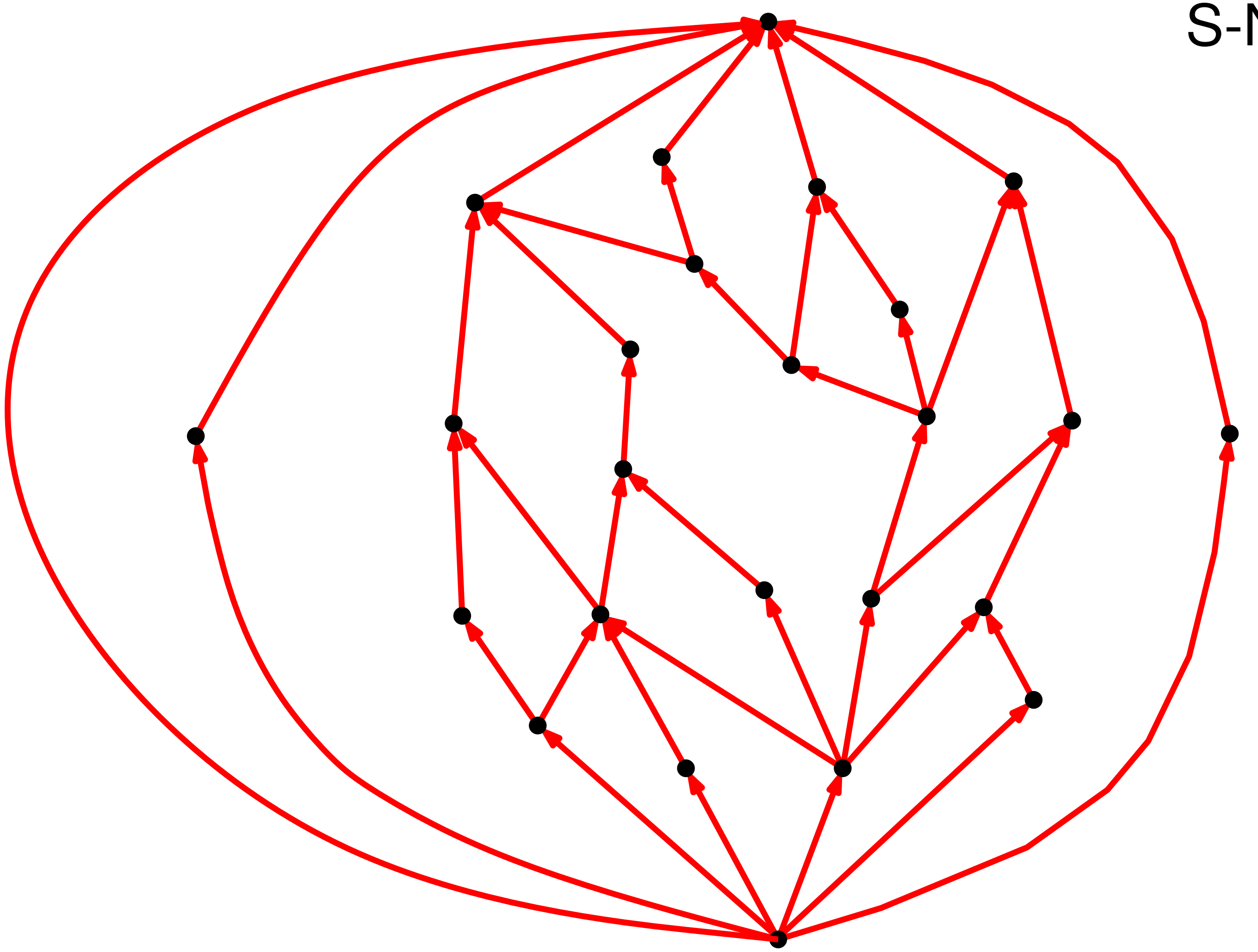


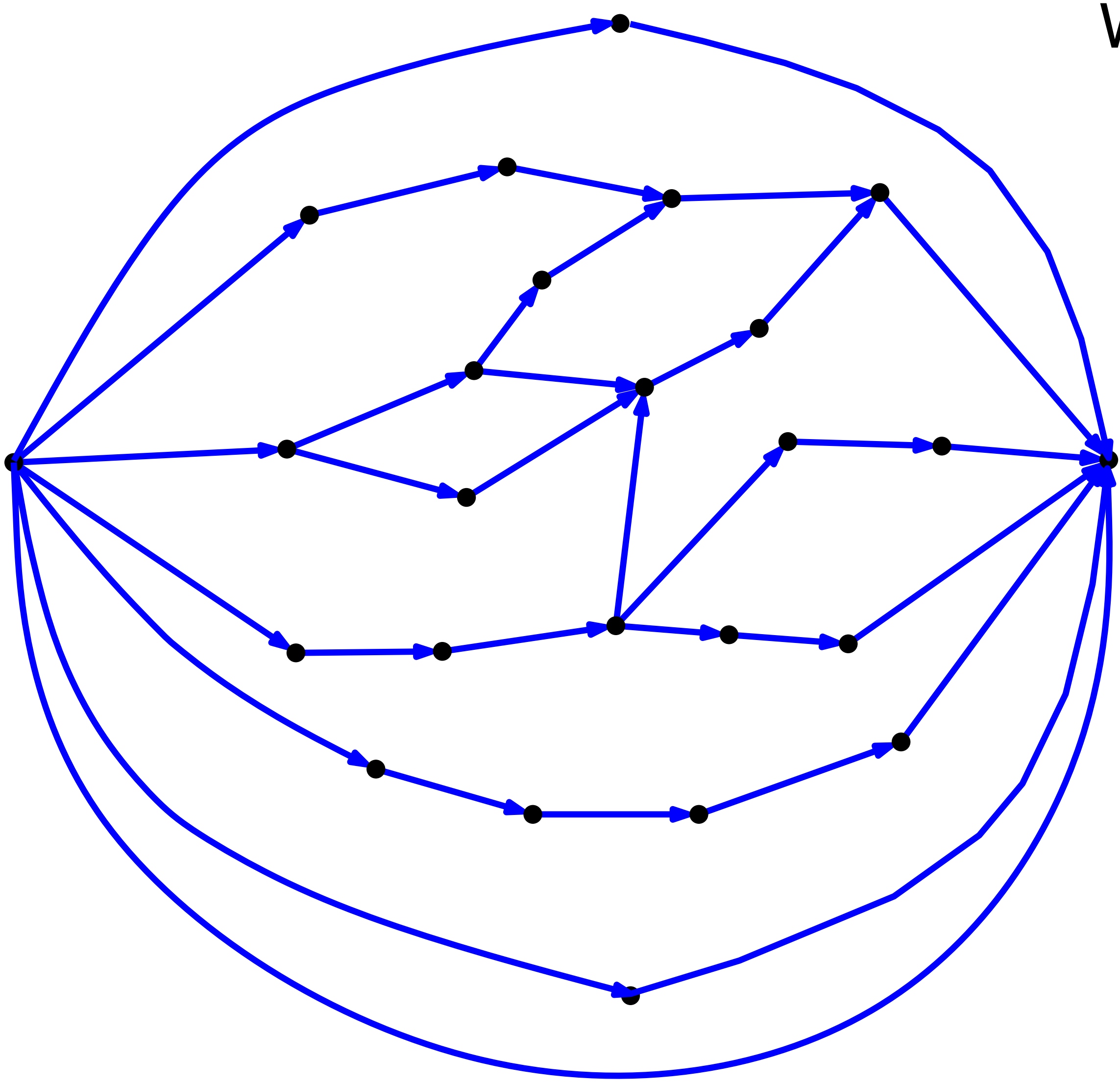
Rectangular Dual

S-N net G_{S-N}



Rectangular Dual

W-E net G_{W-E}



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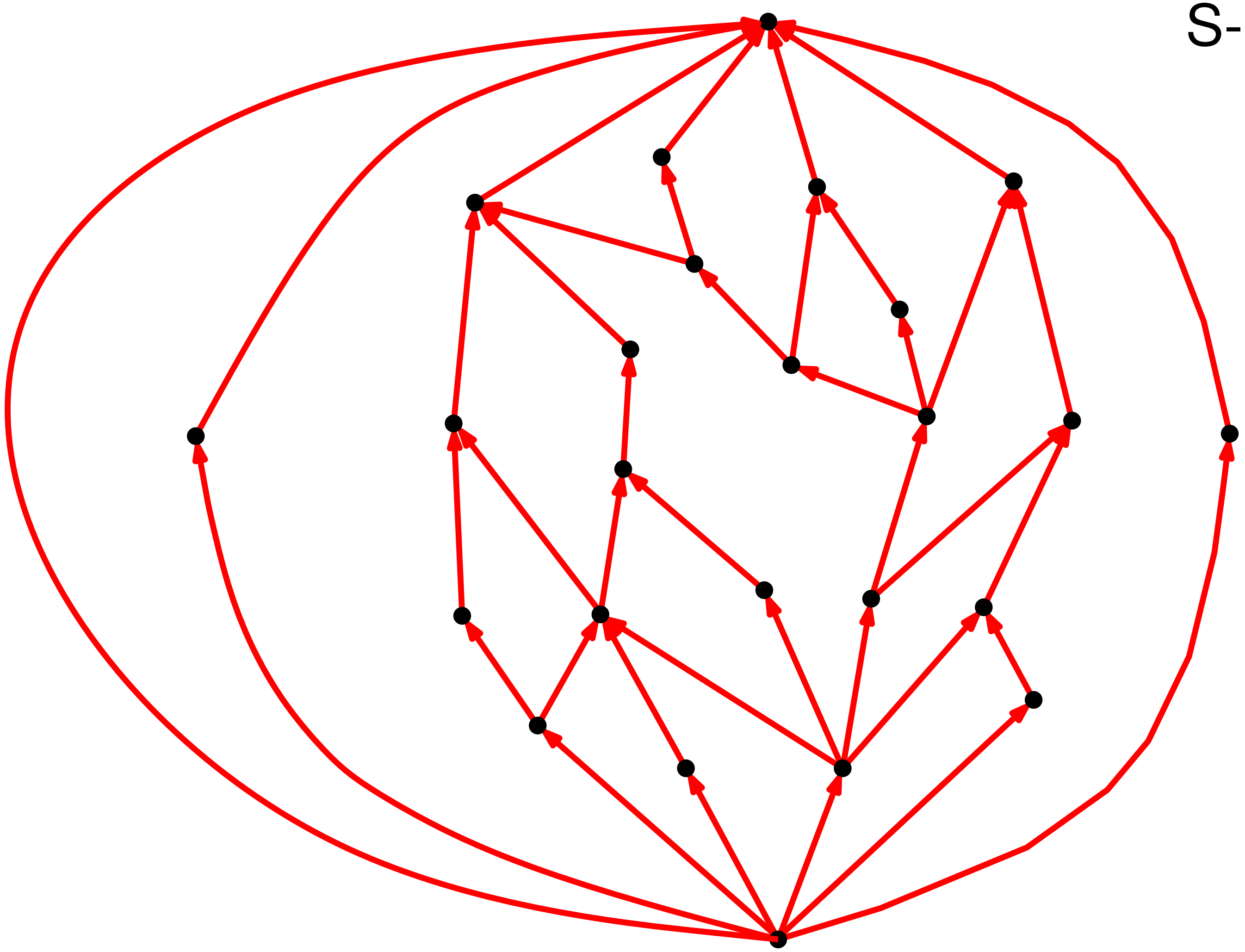

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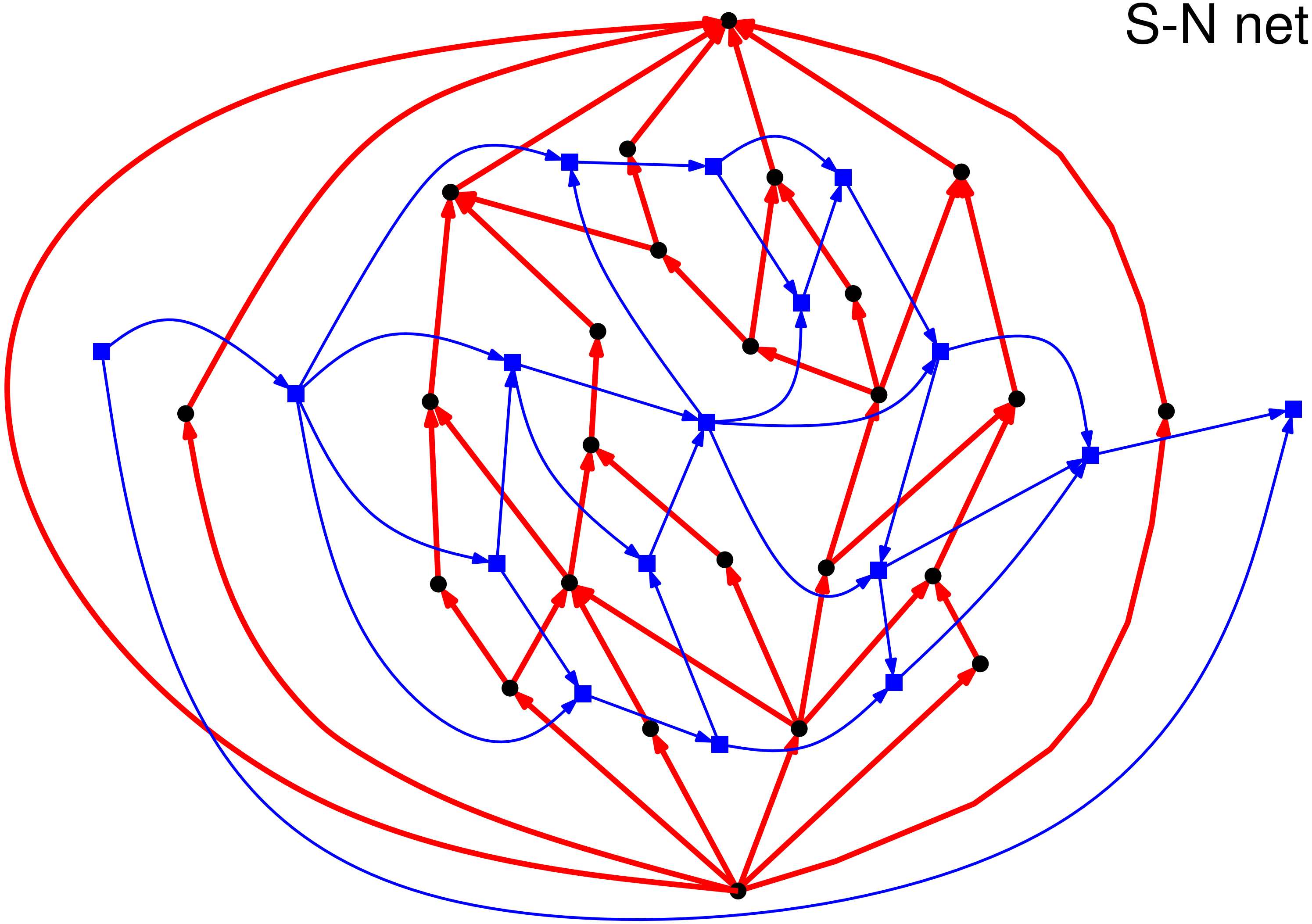

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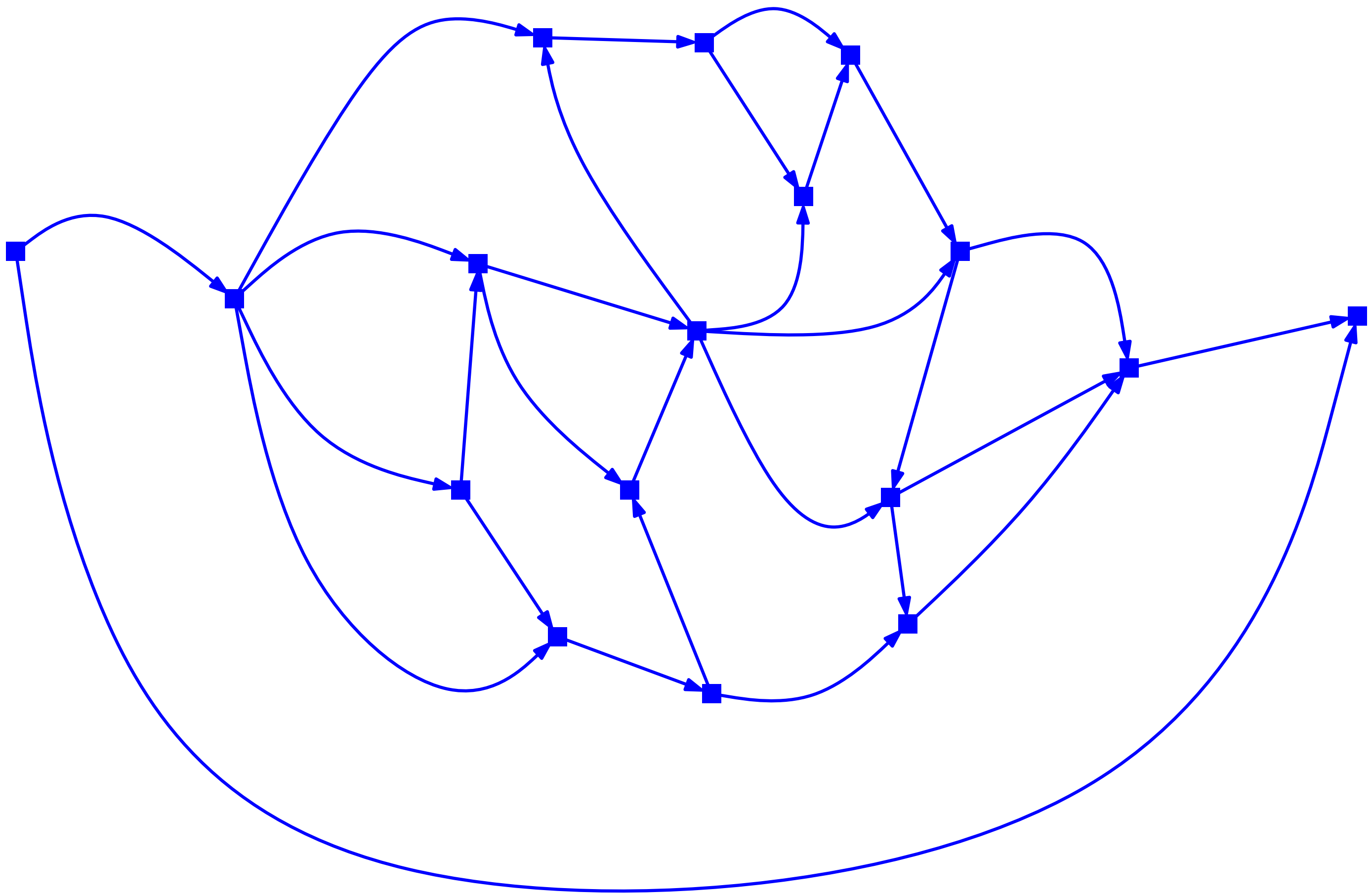

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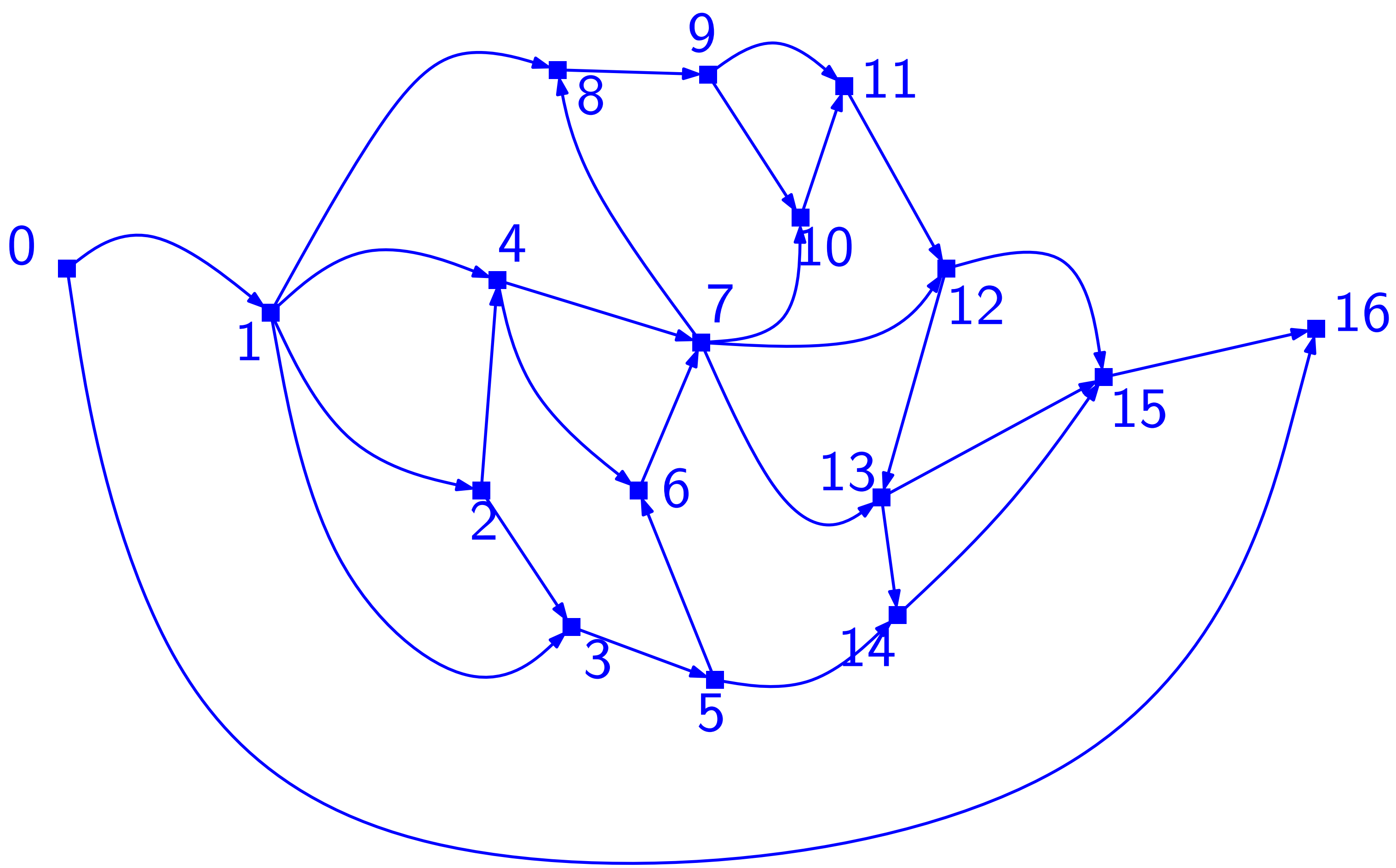

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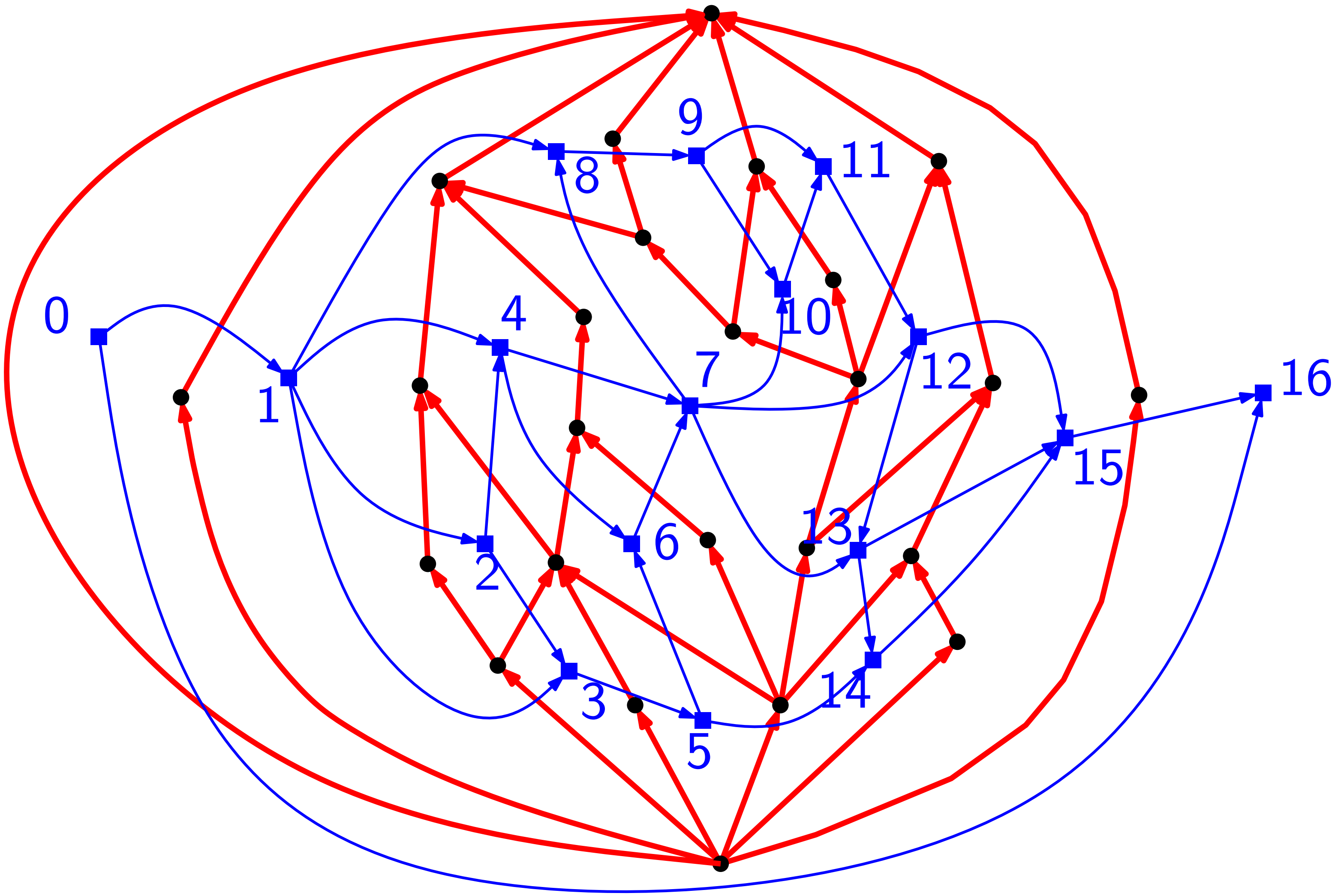

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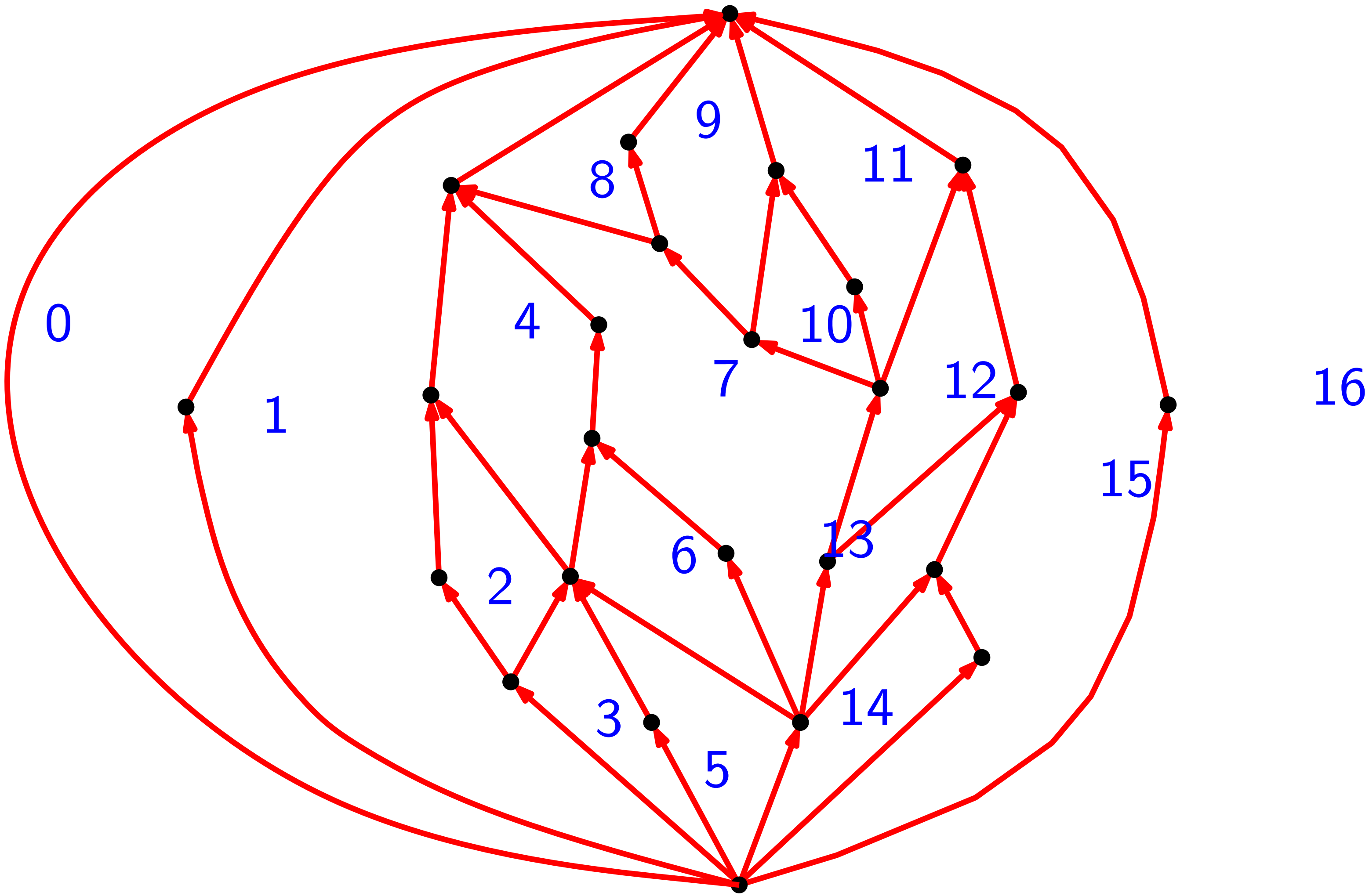

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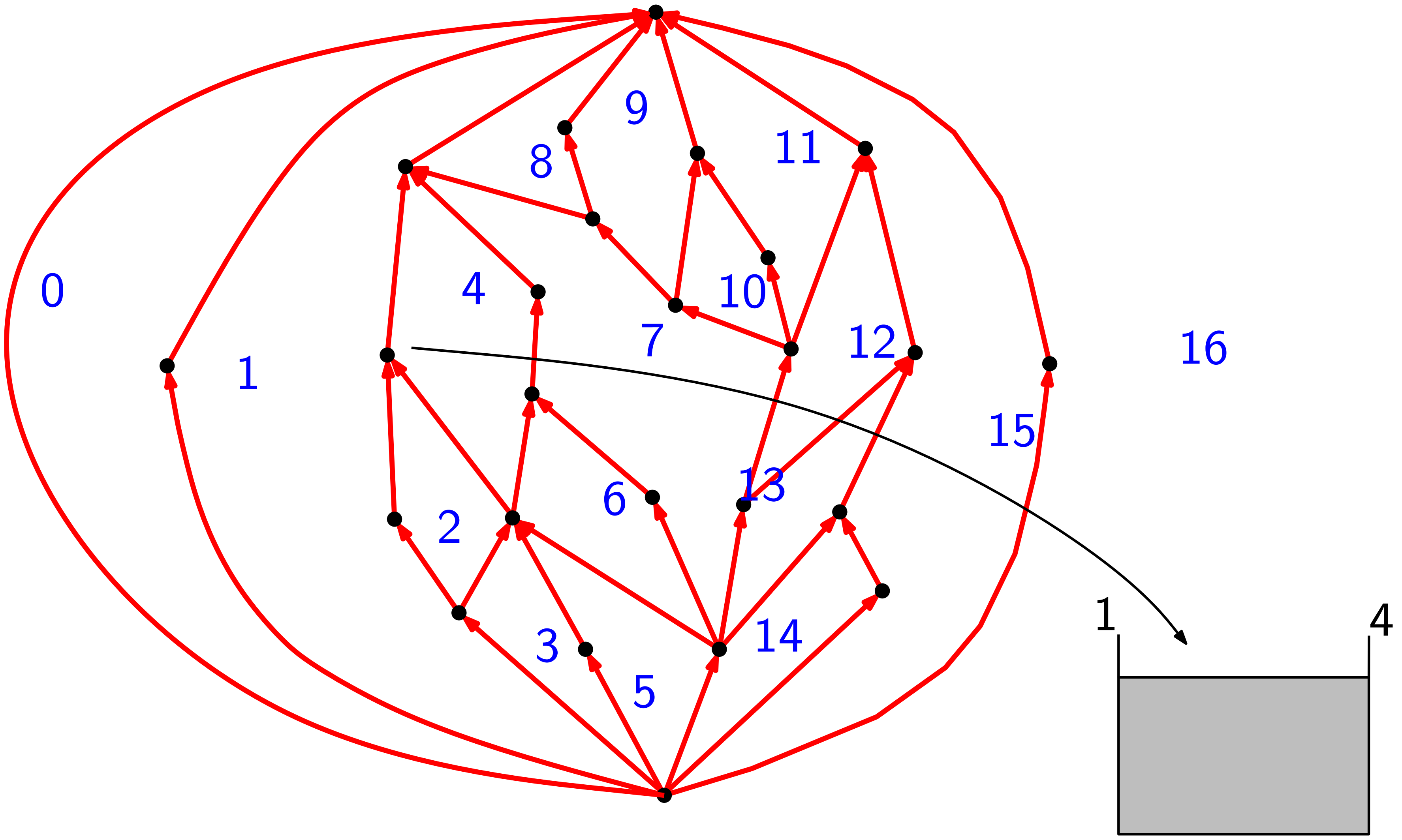

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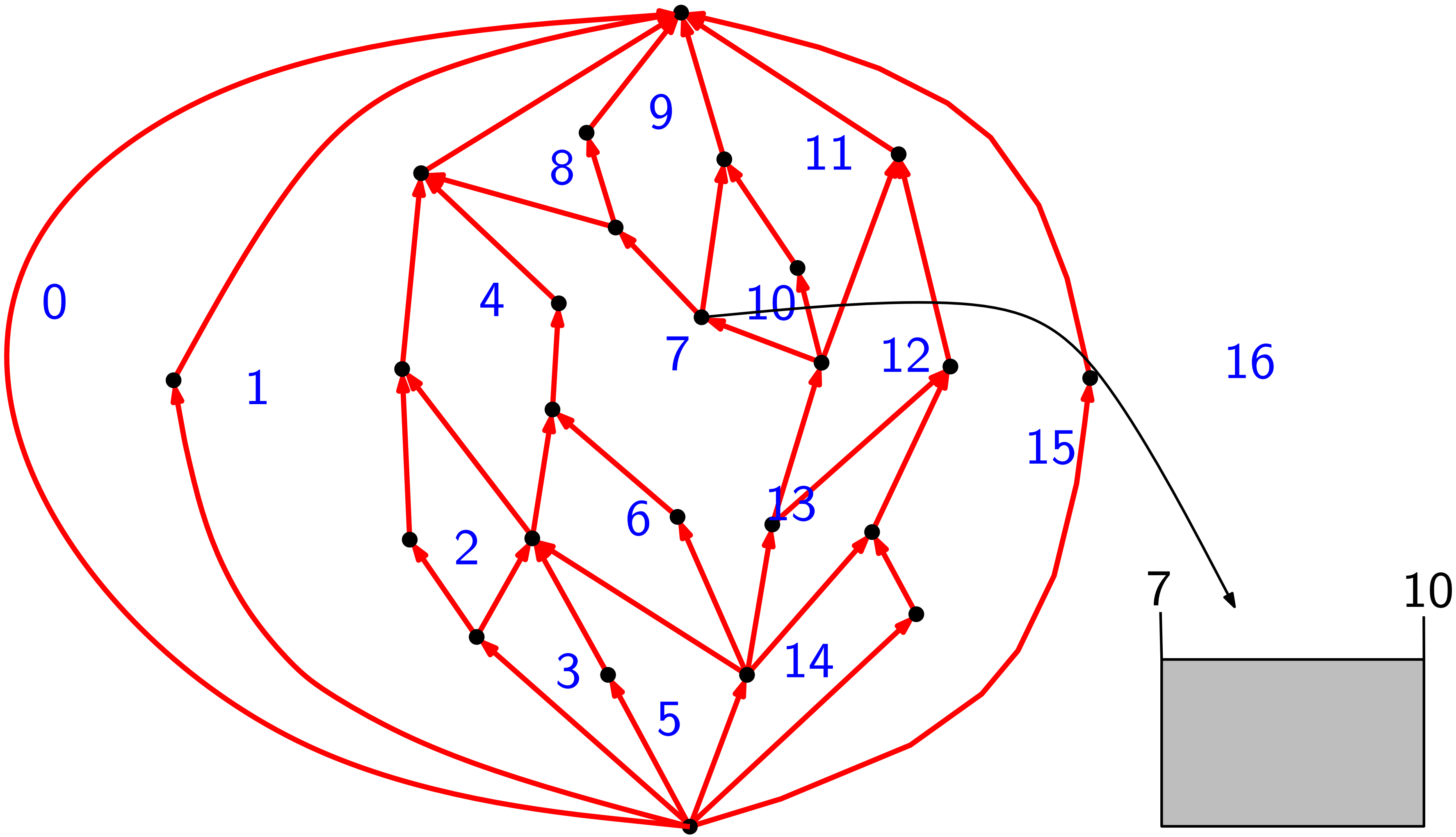

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a S-N net G_{S-N} of G (consists of T_r plus outer edges)
- Construct the dual $G^\star_{S-N}$ of G_{S-N} and compute an st-ordering f_{sn} of $G^\star_{S-N}$
- For each vertex $v \in V$, let f and g be the face on the left and face on the right of v. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$

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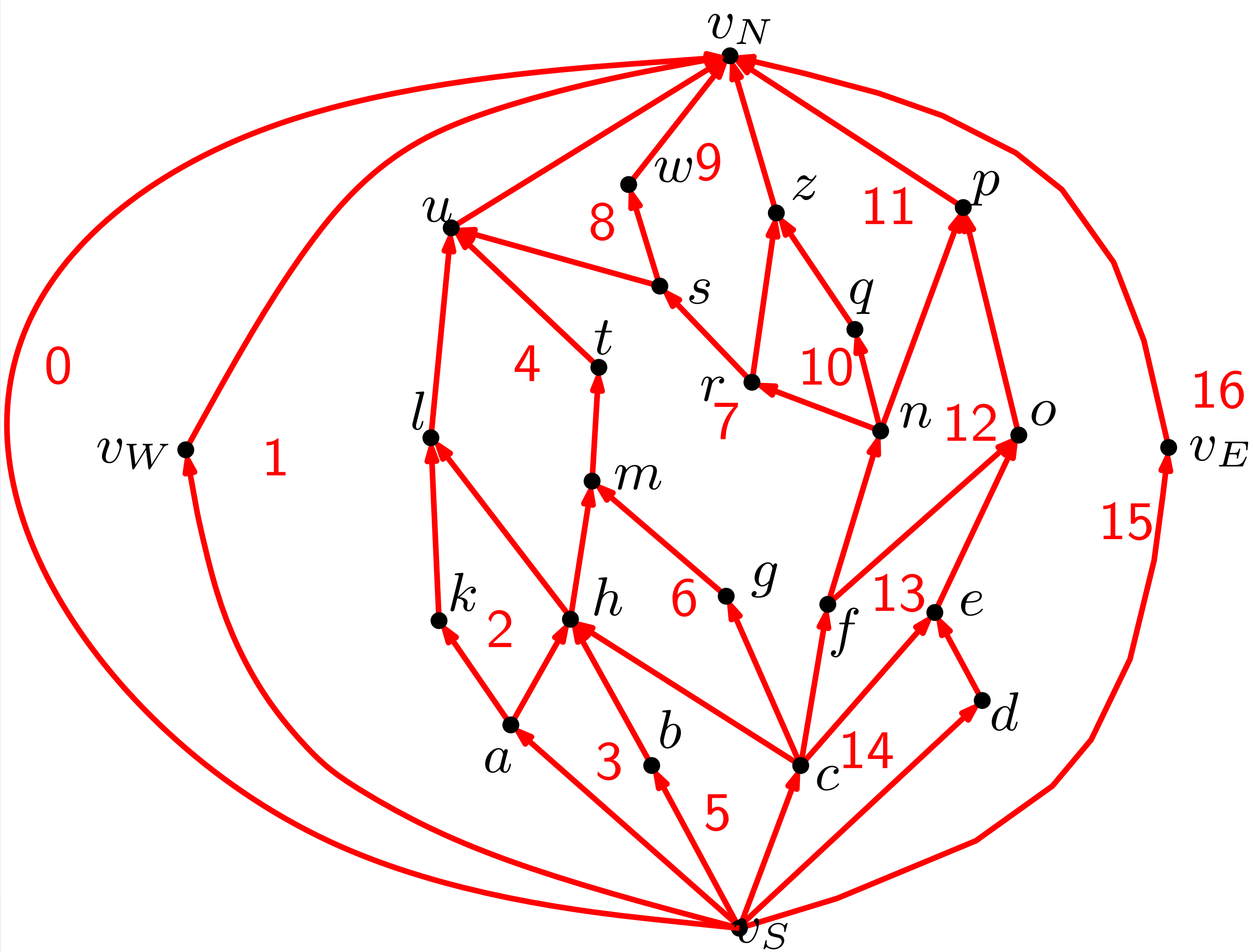
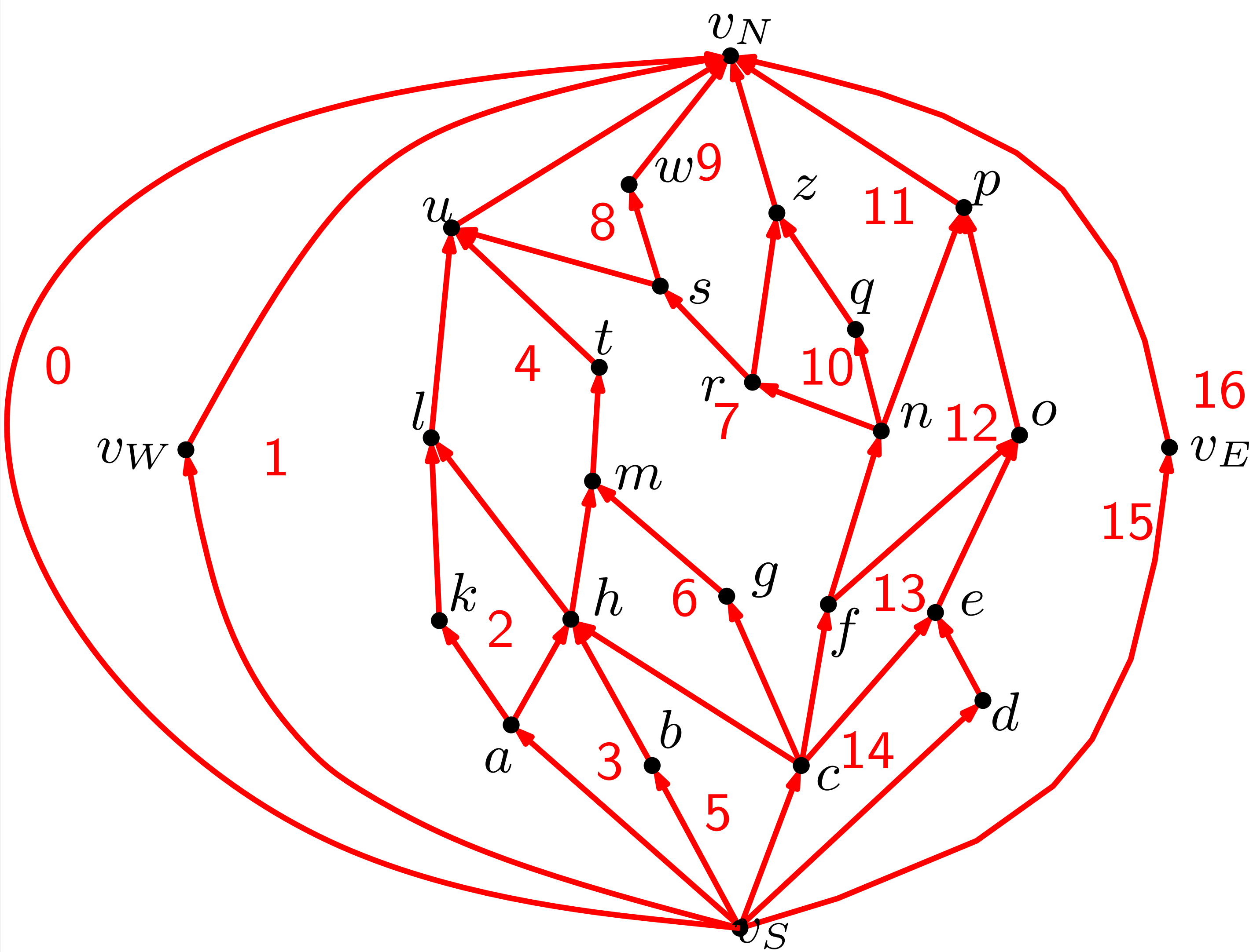
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v),\ x_2(v)$ and y-coordinates $y_1(v),\ y_2(v)$.

Rectangular Dual

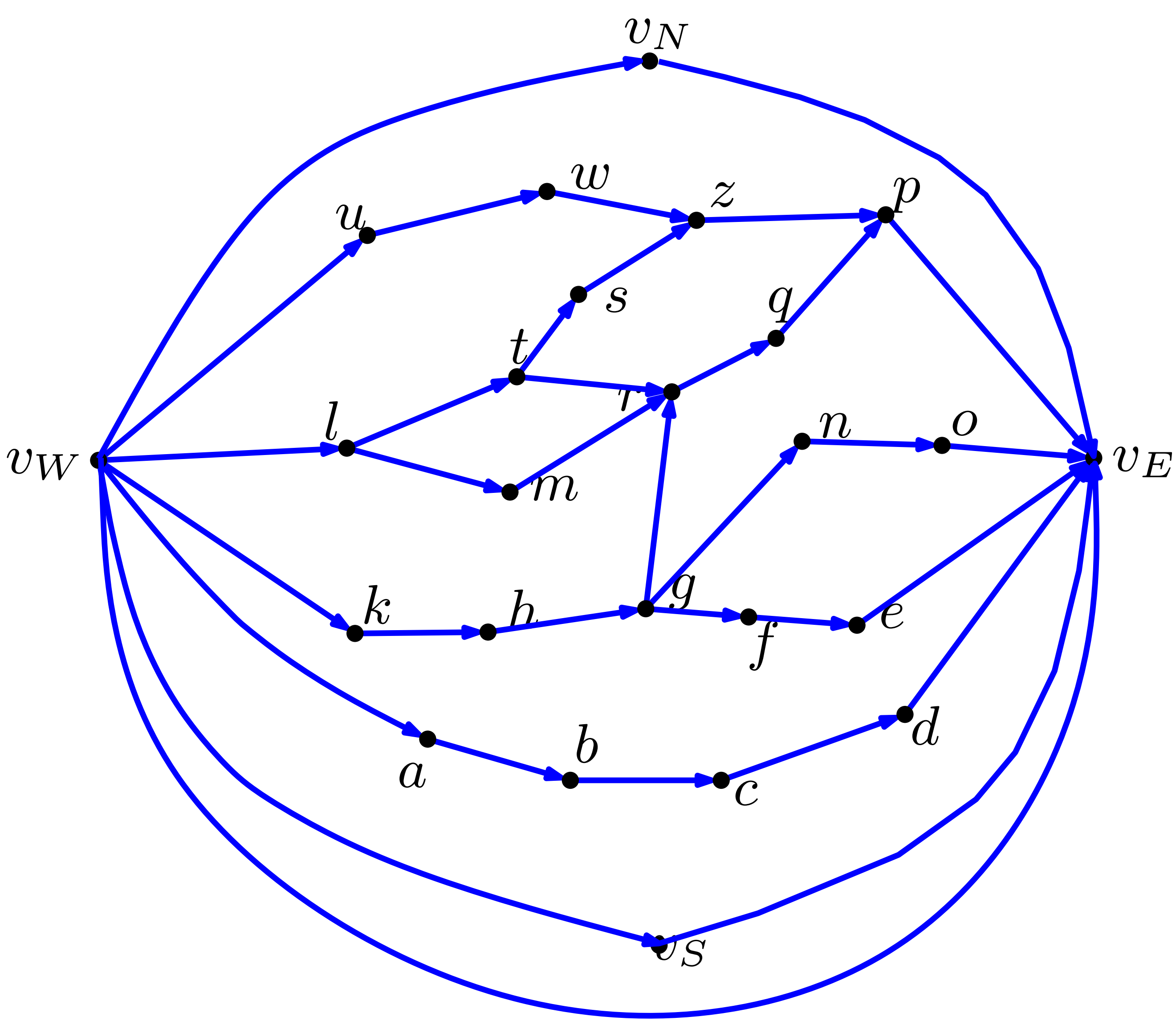


Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



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$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
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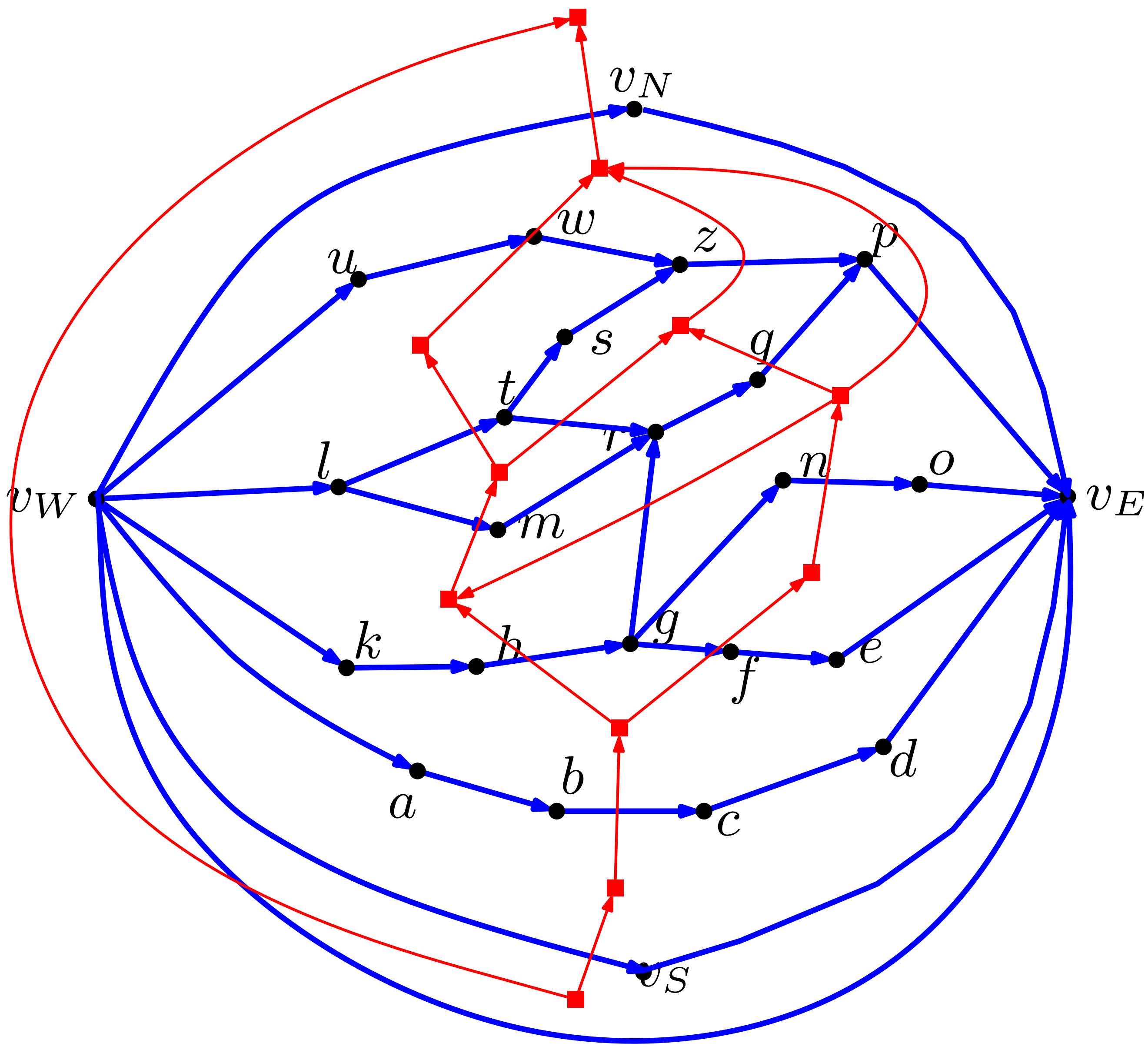
Rectangular Dual



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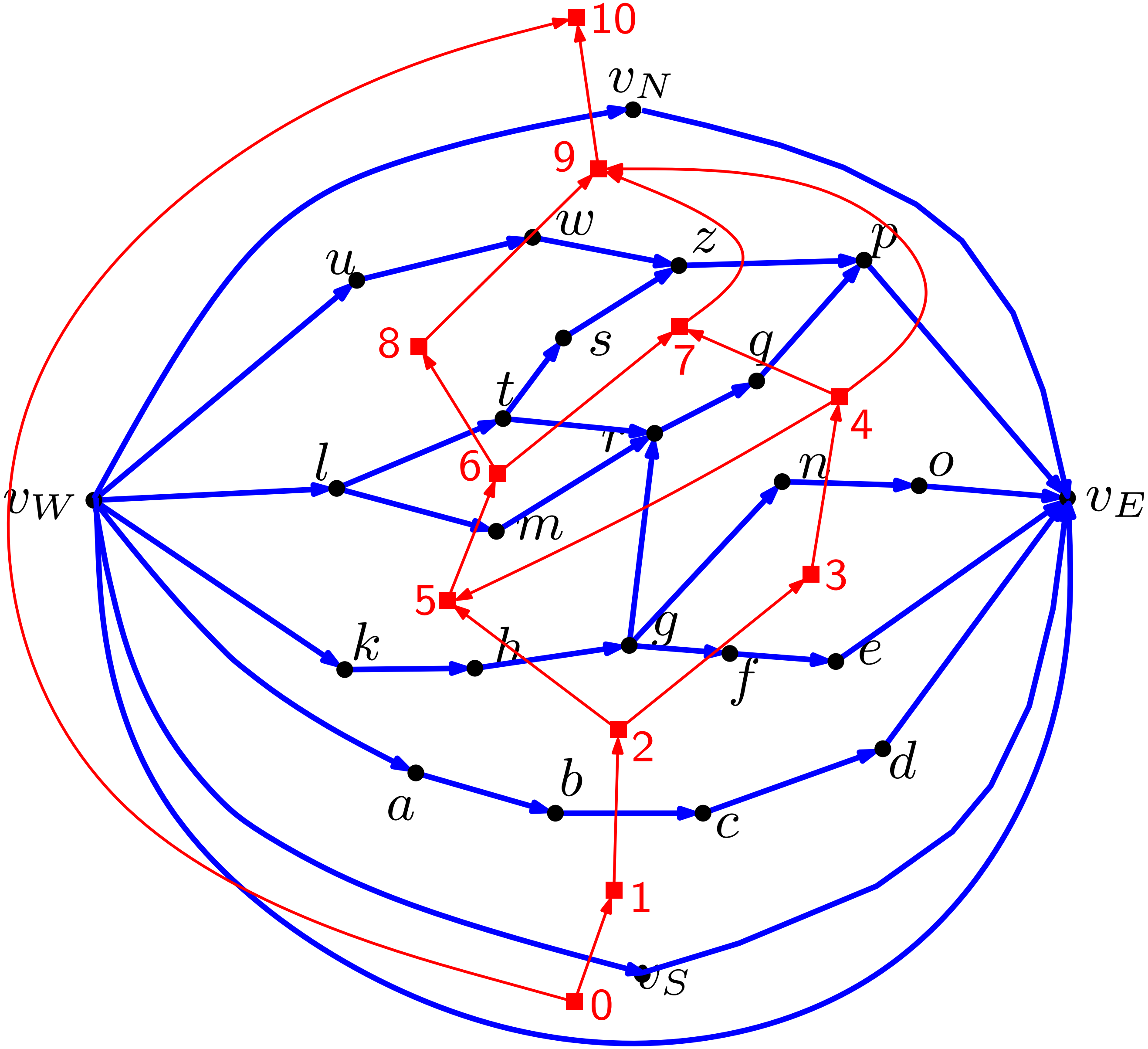
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
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$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
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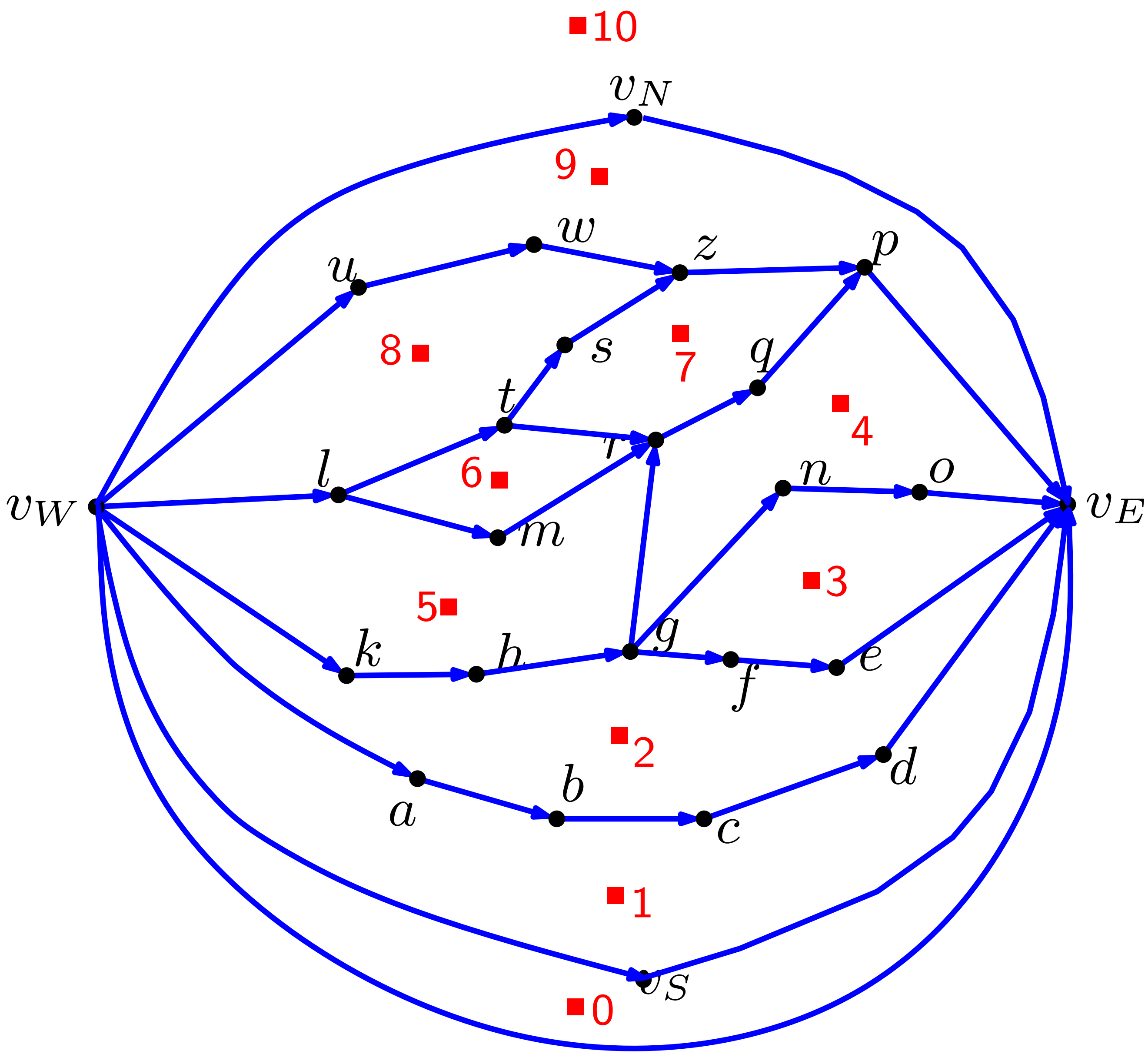
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
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$x_1(a) = 1,\ x_2(a) = 3$
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$x_1(c) = 5,\ x_2(c) = 14$
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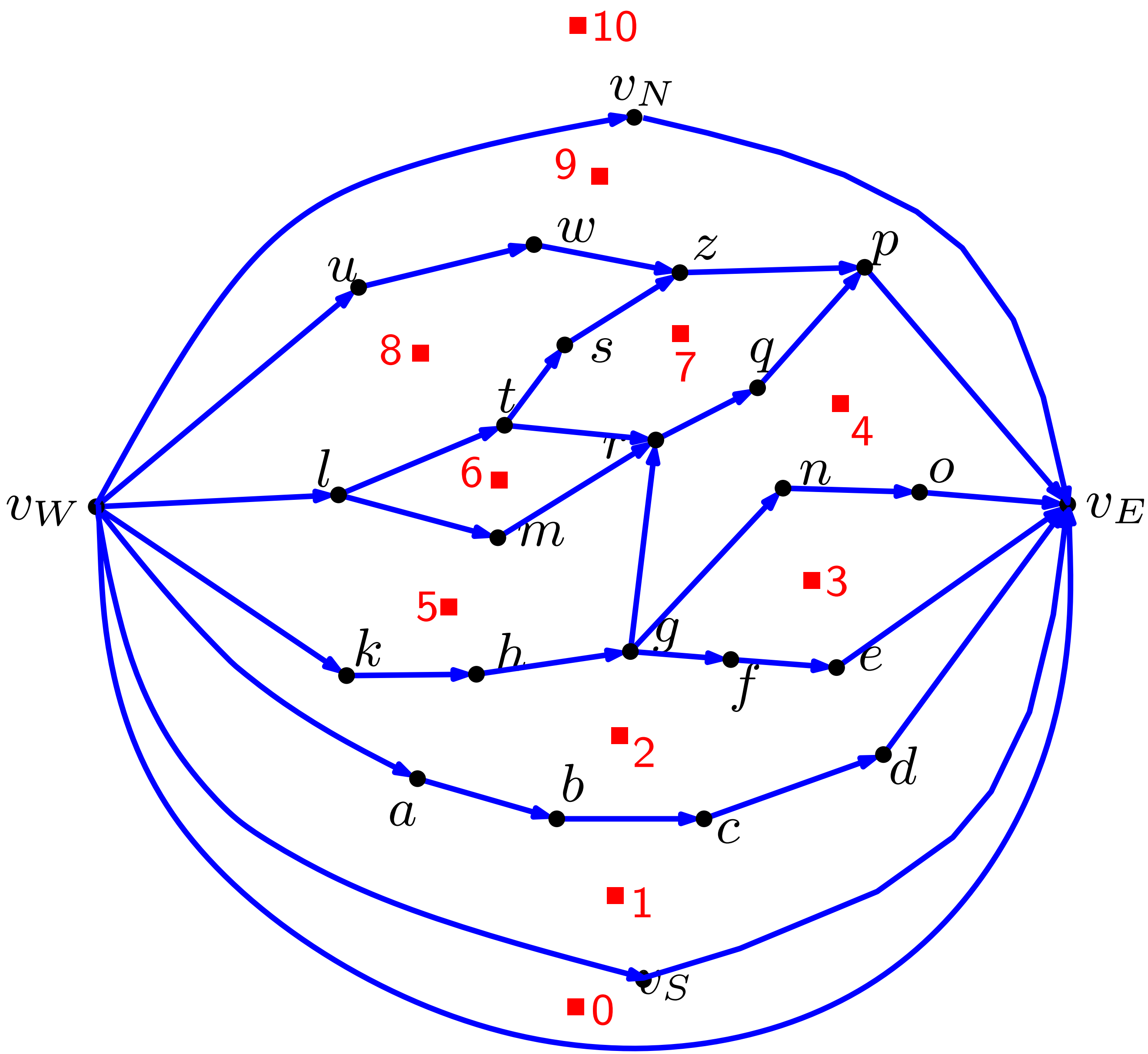
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



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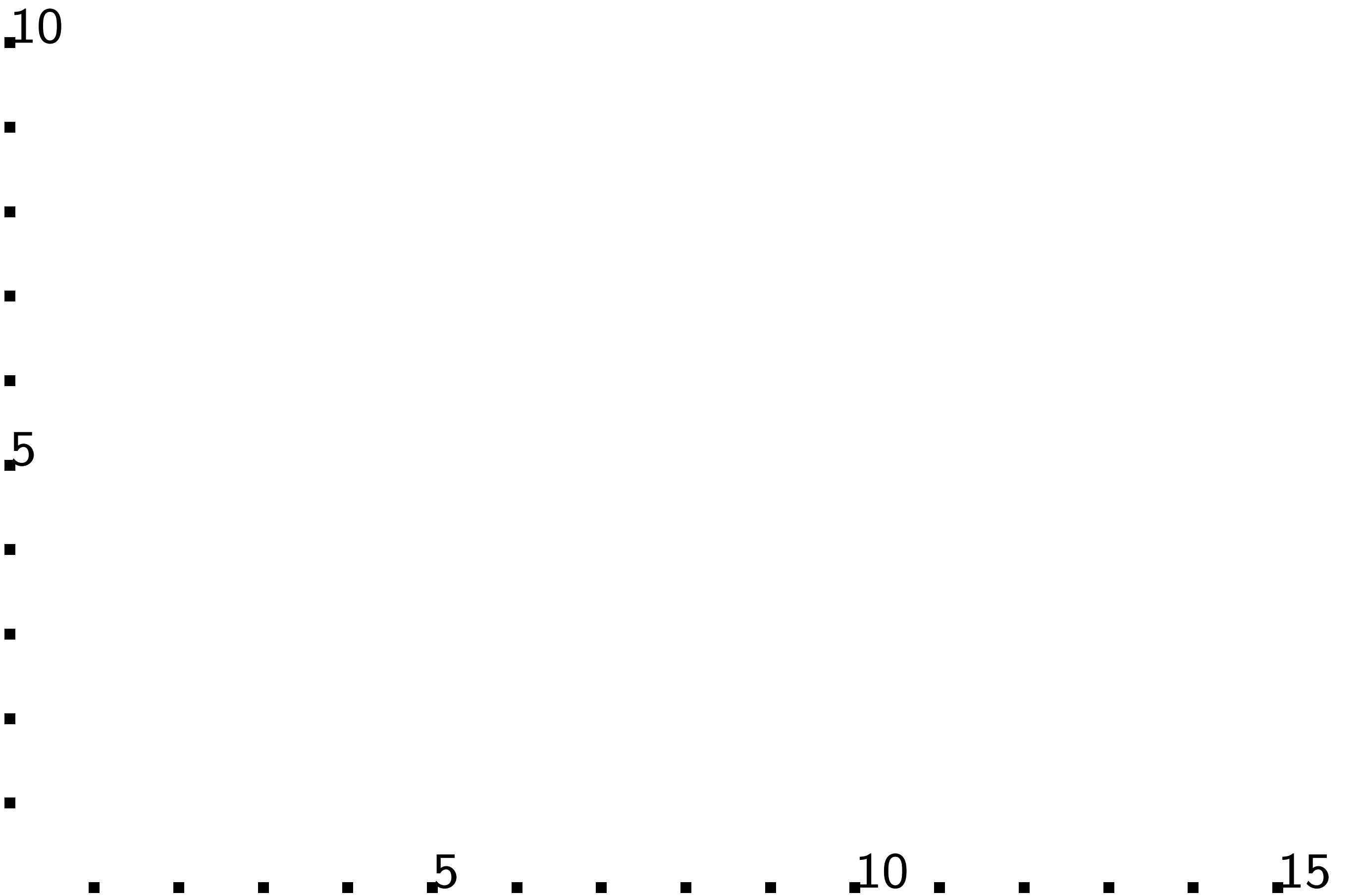
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual



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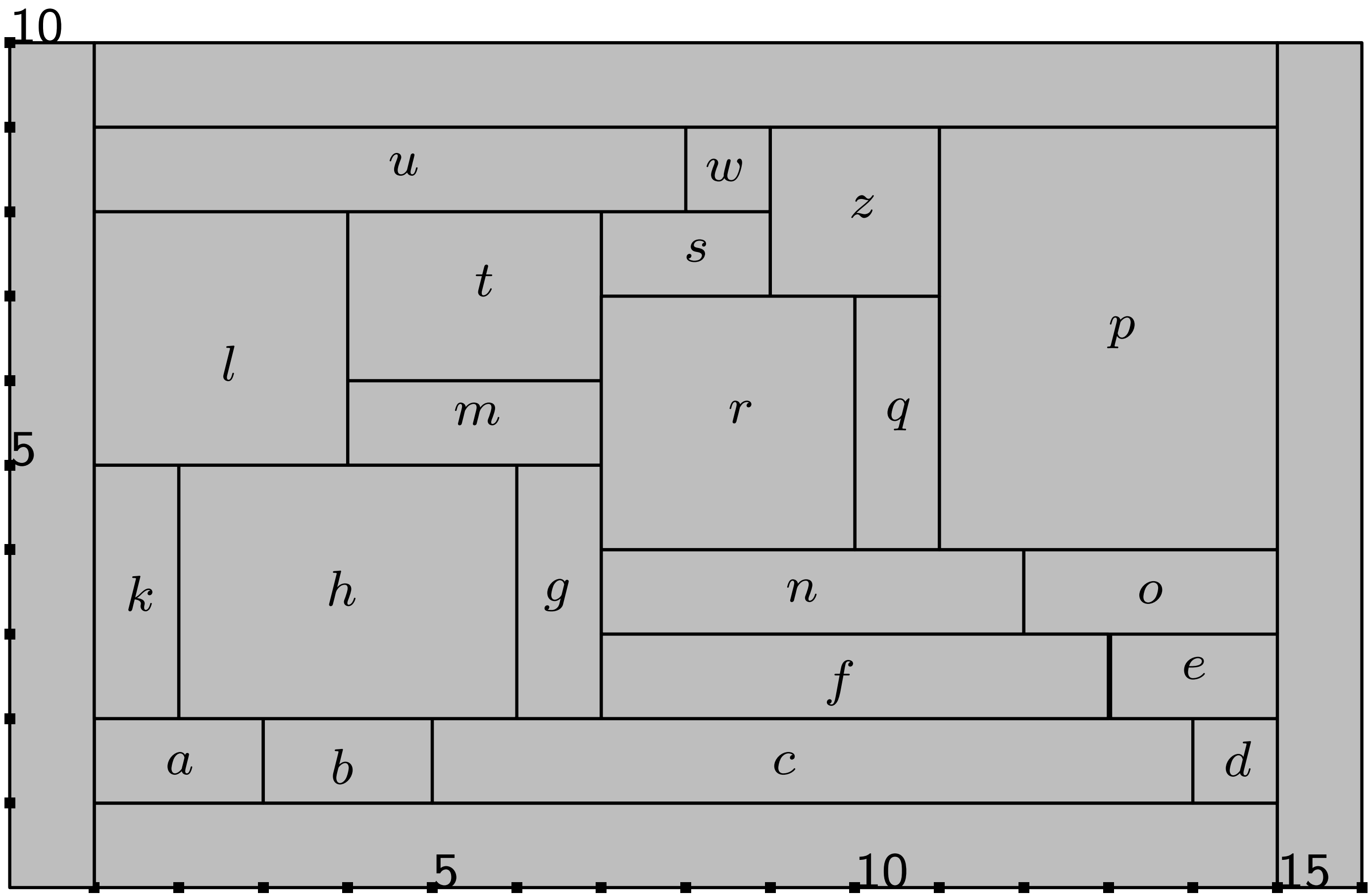
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
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$x_1(a) = 1,\ x_2(a) = 3$
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$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

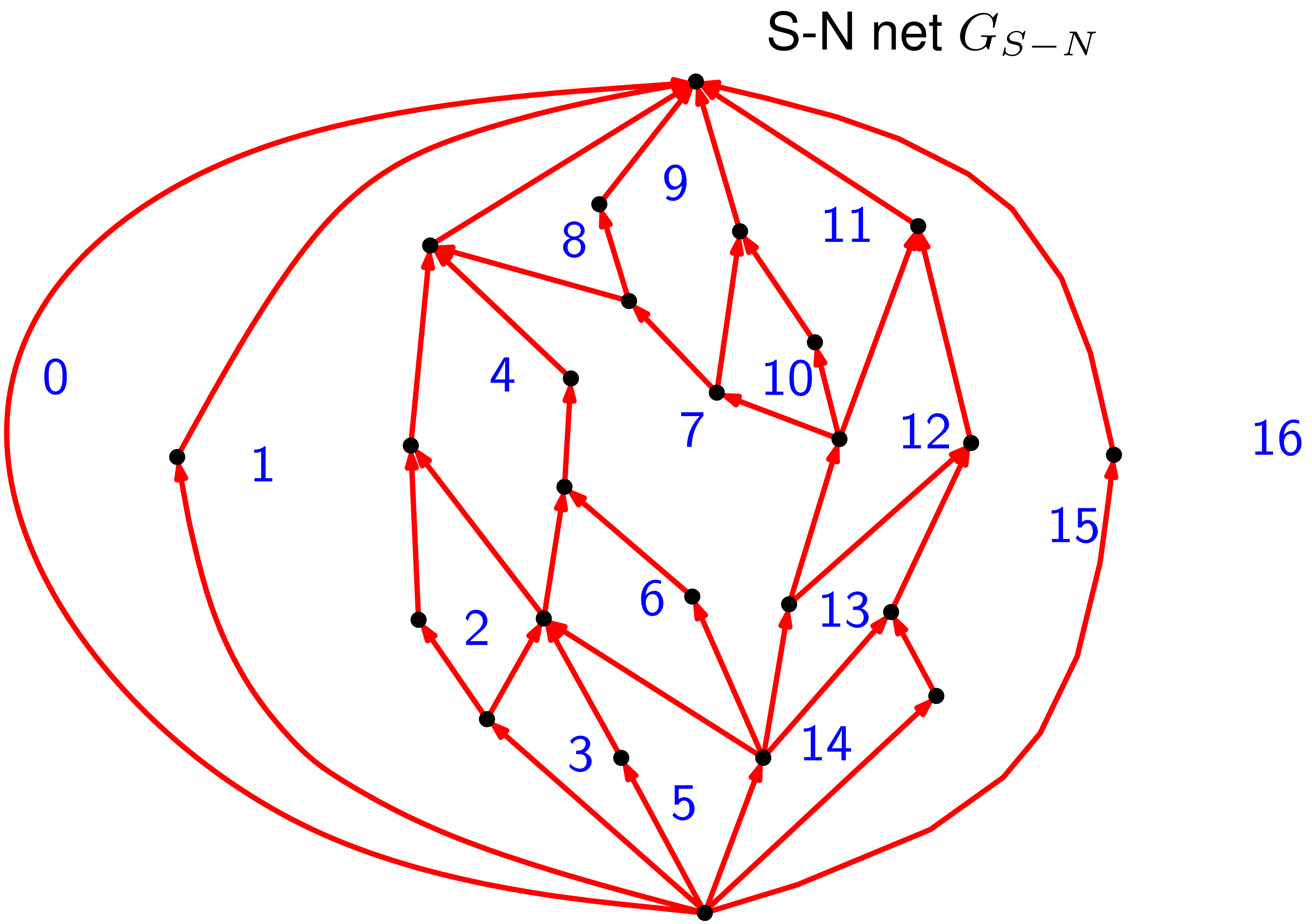
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual

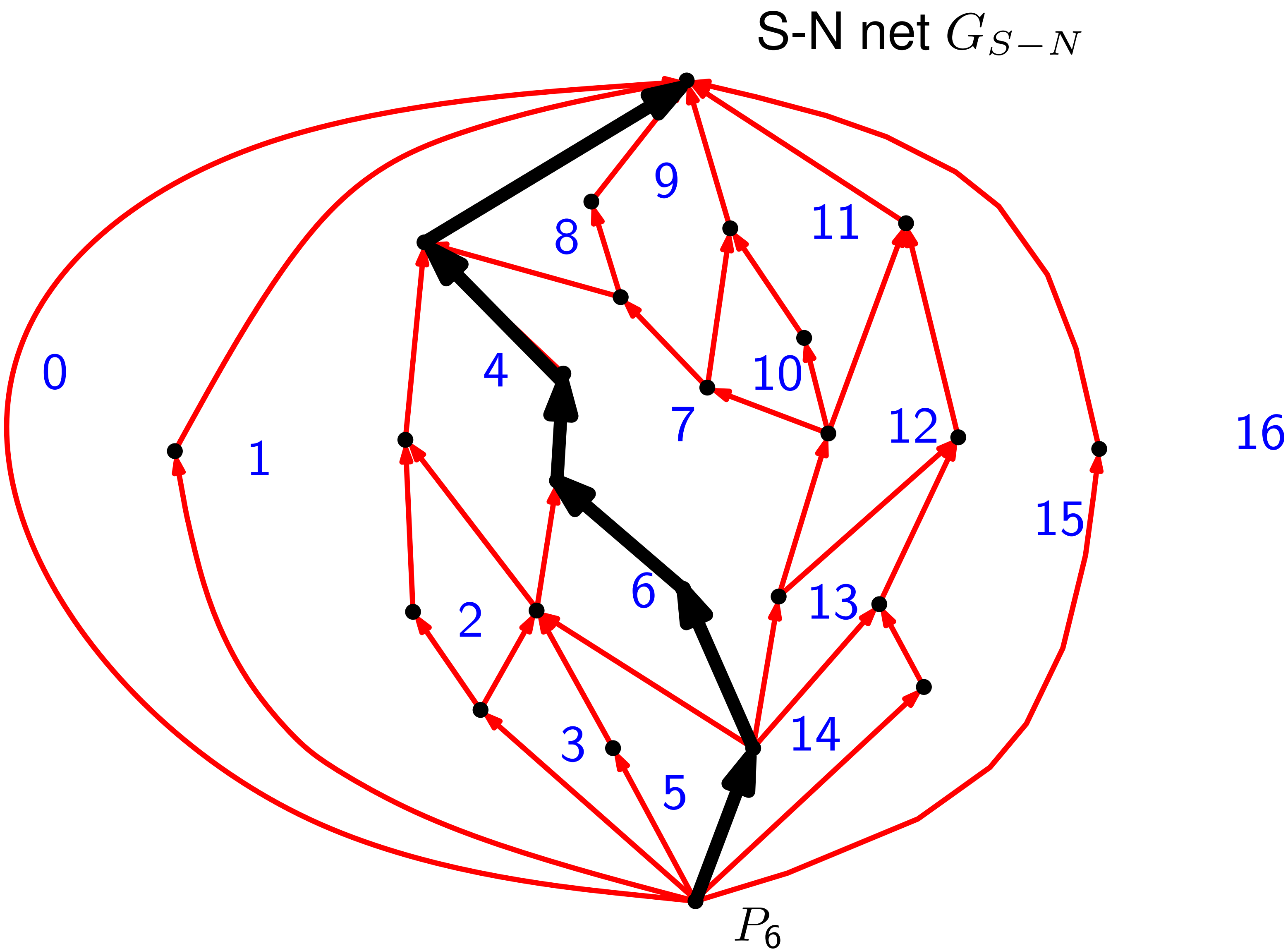
In the following we prove that presented algorithm constructs a rectangular dual of G.

- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) be the dual of G_{S-N} (resp. G_{W-E})
- Let $f_1, \ldots, f_k$ be the faces of $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) , enumerated according to st-numbering f_{sn} (resp. f_{we})
- Let G^{i}_{S-N} (resp. G^{i}_{W-E}) denote the subgraph of G that is induced by vertices and edges of $f_1, \ldots, f_i$
- We denote P_i (reps. Q_i) the right (resp. top) boundary of G^{i}_{S-N} (resp. G^{i}_{W-E}) .

Rectangular Dual

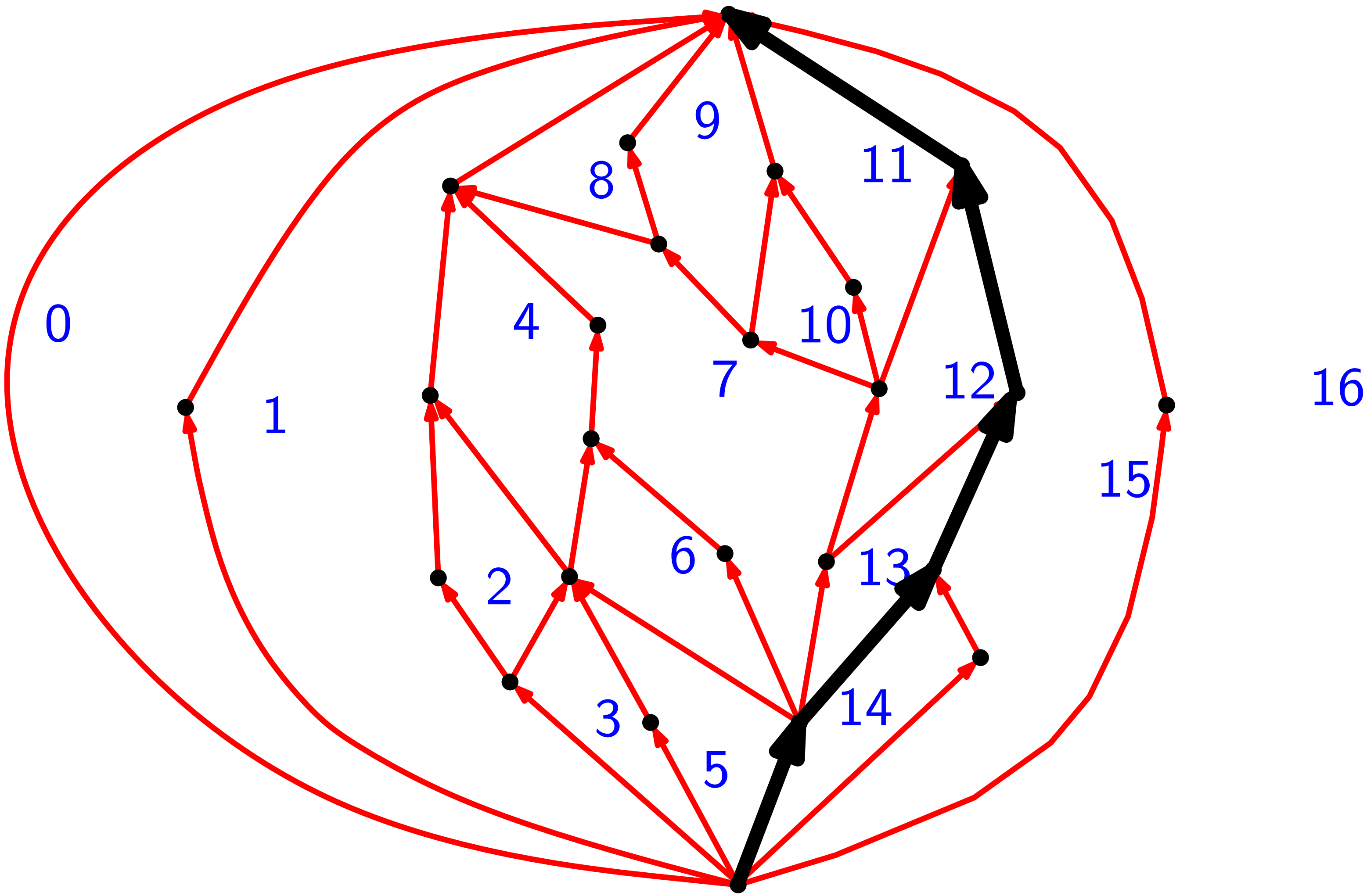


Rectangular Dual

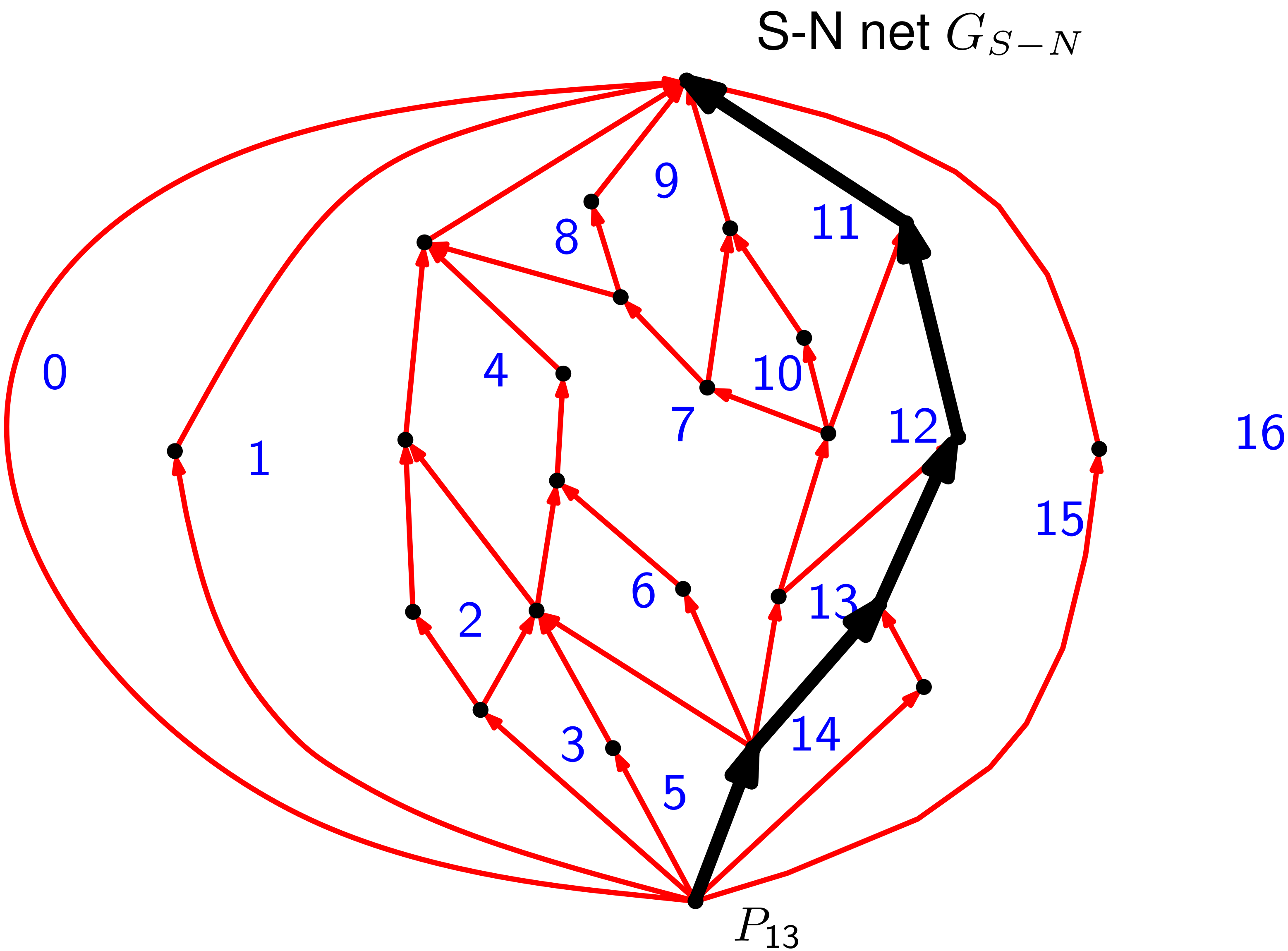


Rectangular Dual

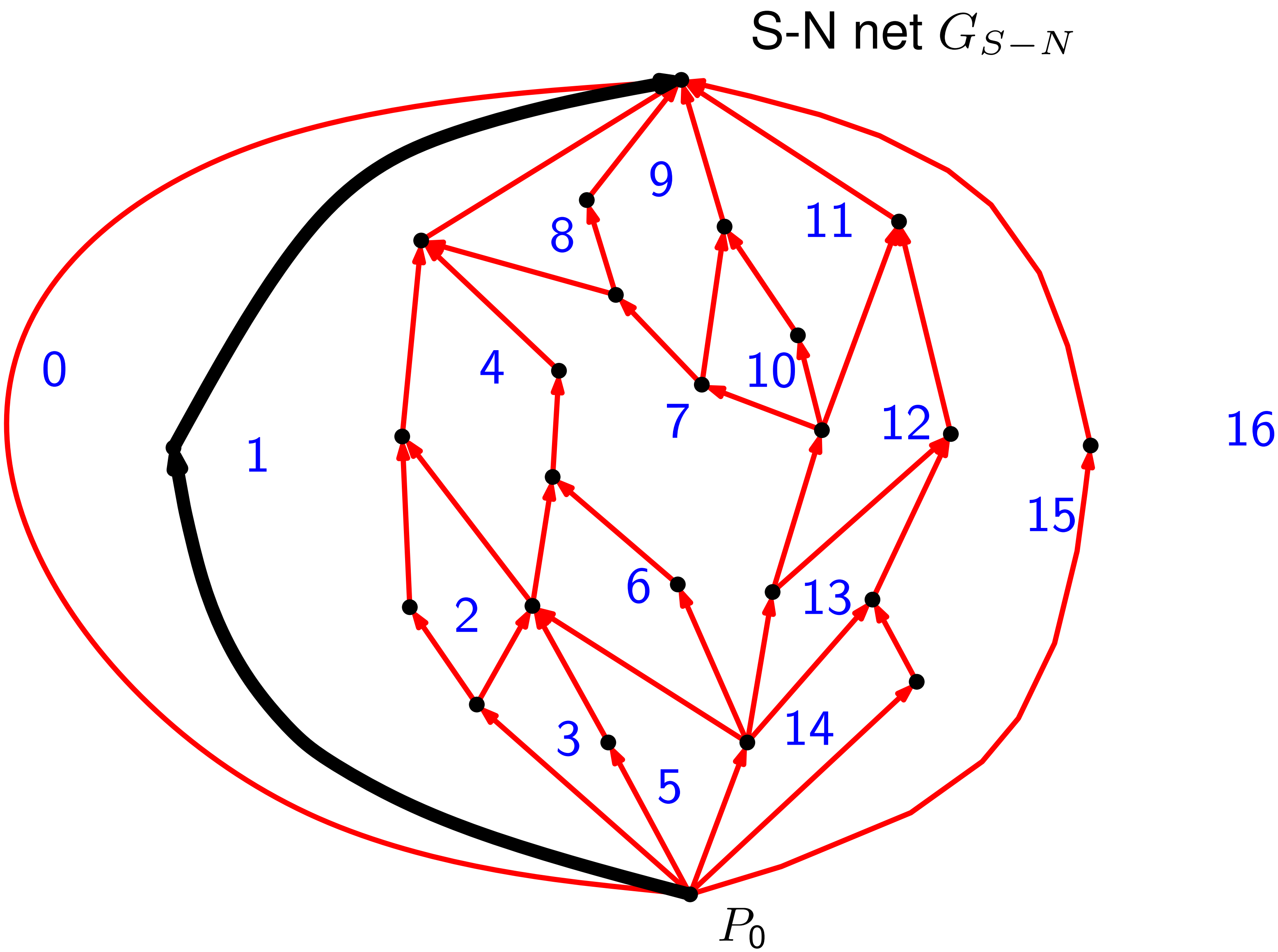
S-N net G_{S-N}



Rectangular Dual



Rectangular Dual



Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

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Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof...

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

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Proof...

Lemma 5

Let $v \in V$, f and g are the fases below and above v in G_{W-E}. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex v belongs to path Q_j if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Rectangular Dual

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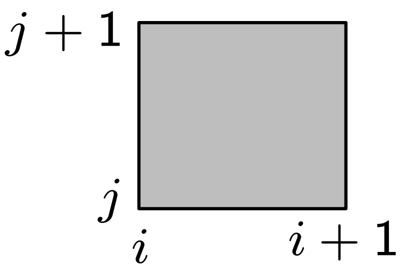
The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

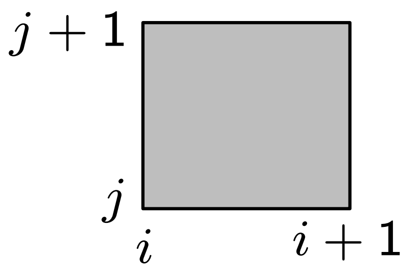


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there **exists a vertex** over this box: $u \in P_i \cup Q_j$

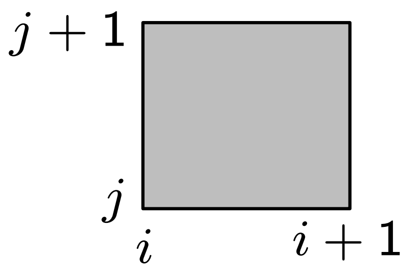


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



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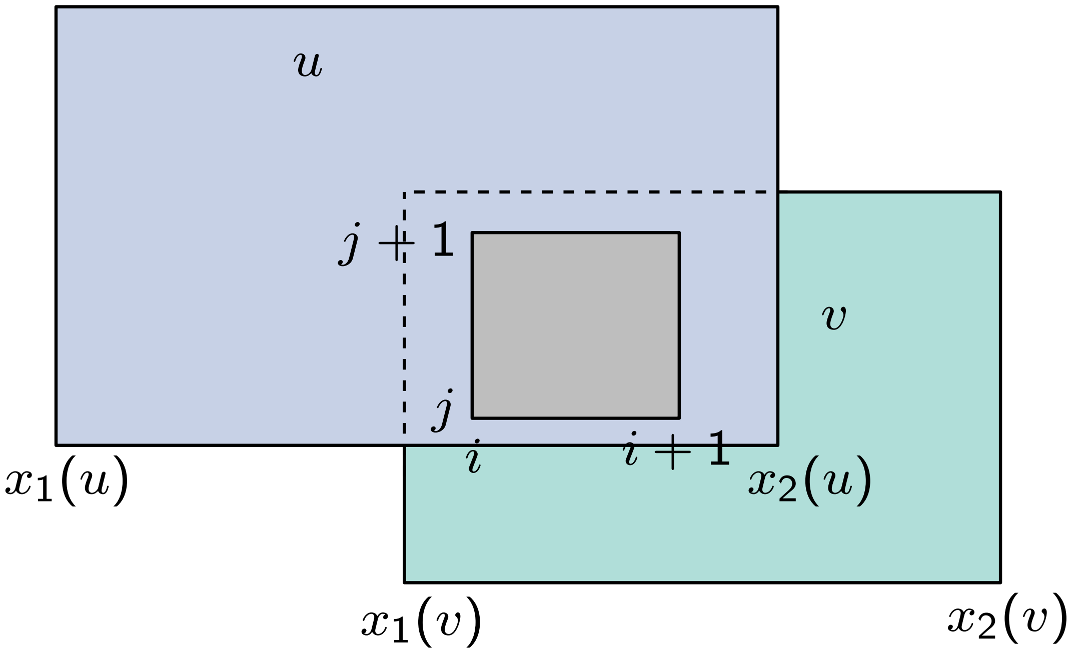

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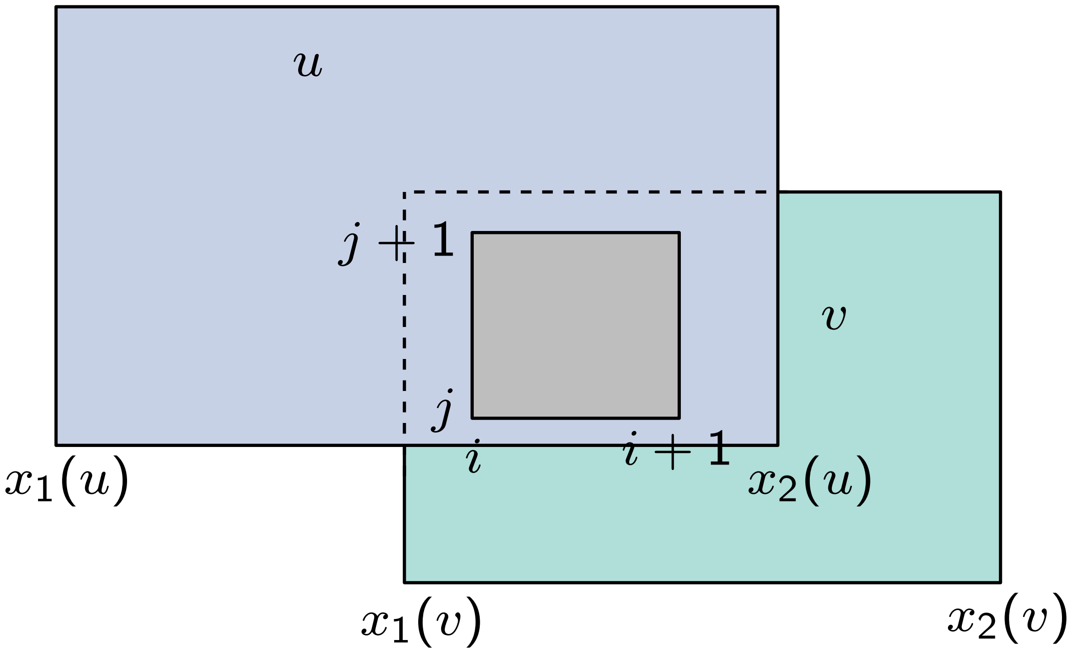

$x_1(u) \leq i$ and $i{+}1 \leq x_2(u)$

Rectangular Dual

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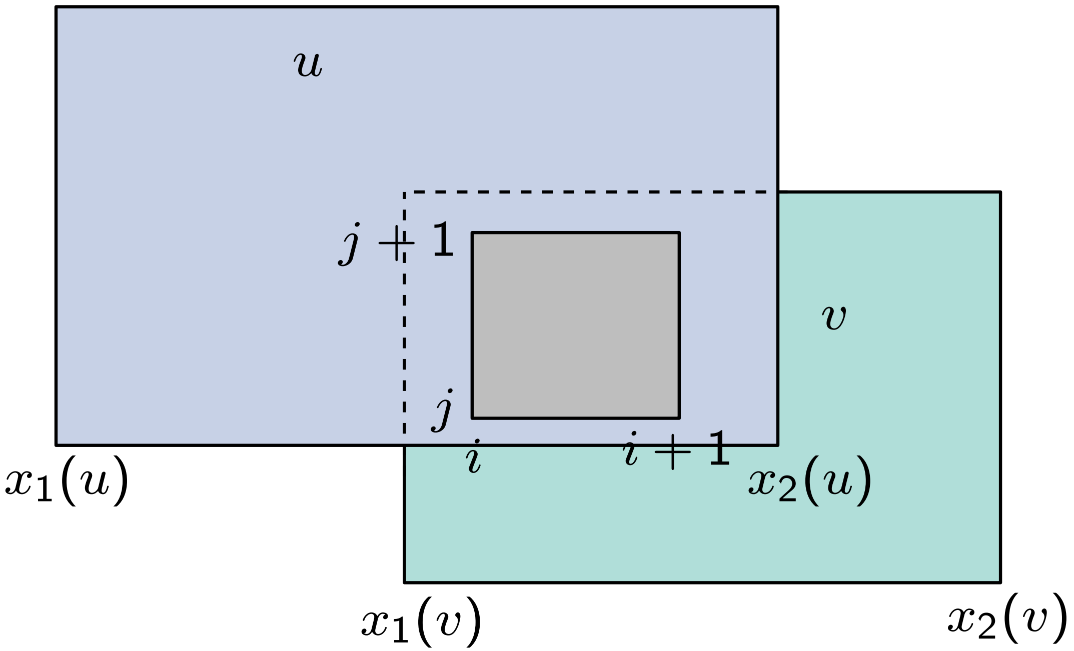

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$\Downarrow$ (Lemma 4)

v belongs to P_i

Rectangular Dual

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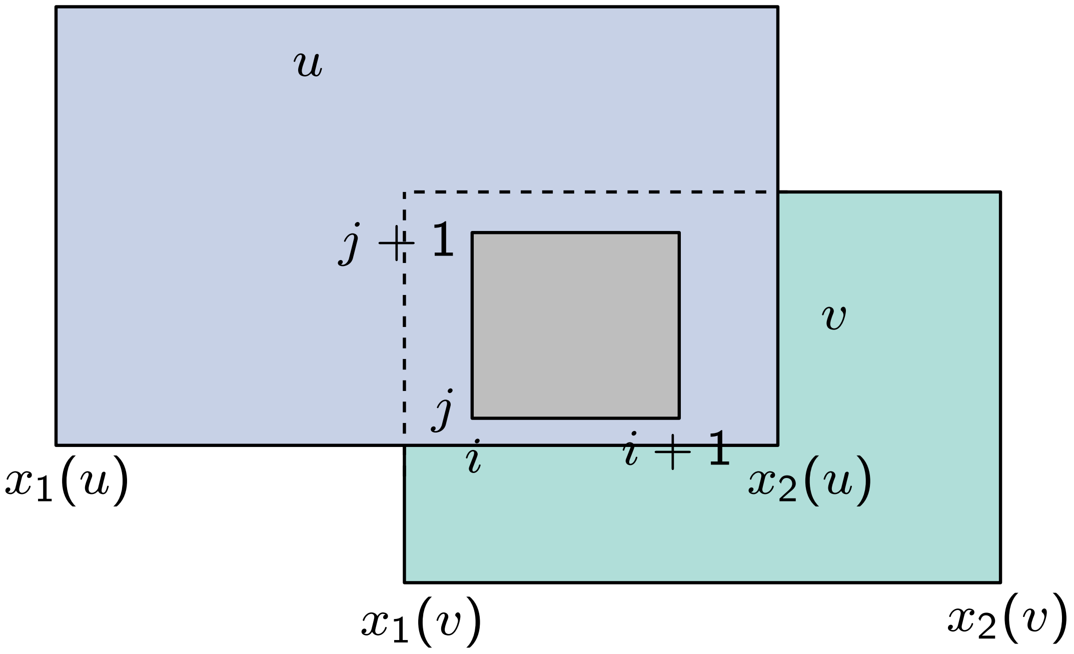

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Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Rectangular Dual

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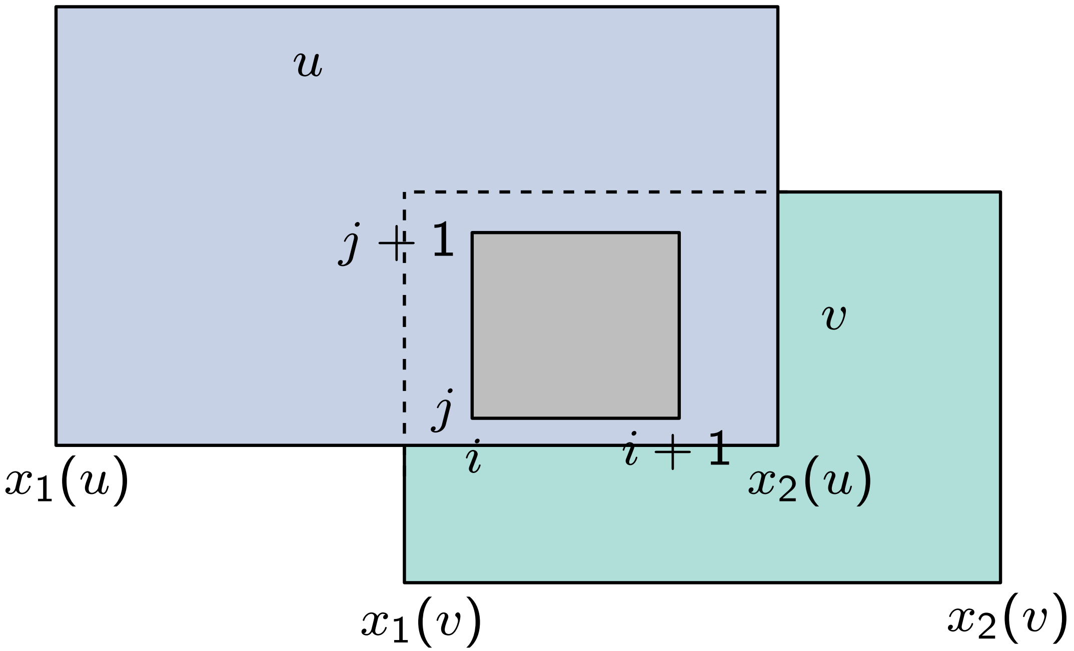

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Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

Rectangular Dual

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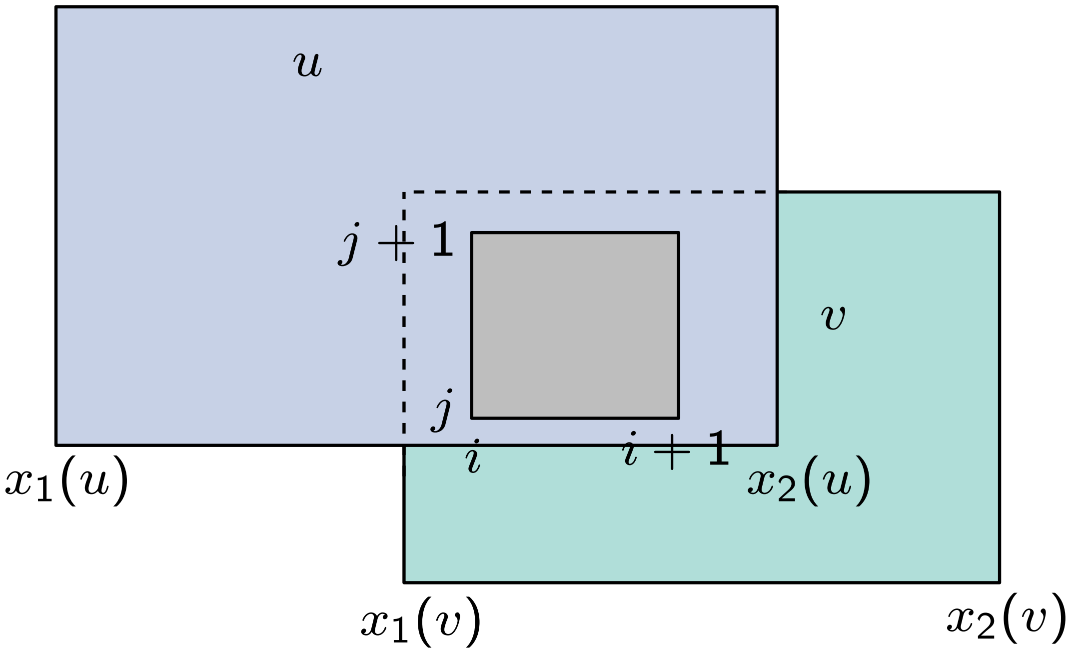

$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

v belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

Which is a contradiction to the property of paths P_i, Q_j except for the cases when: (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$ (corner boxes).

Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

Proof...

Rectangular Dual

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- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

Proof...

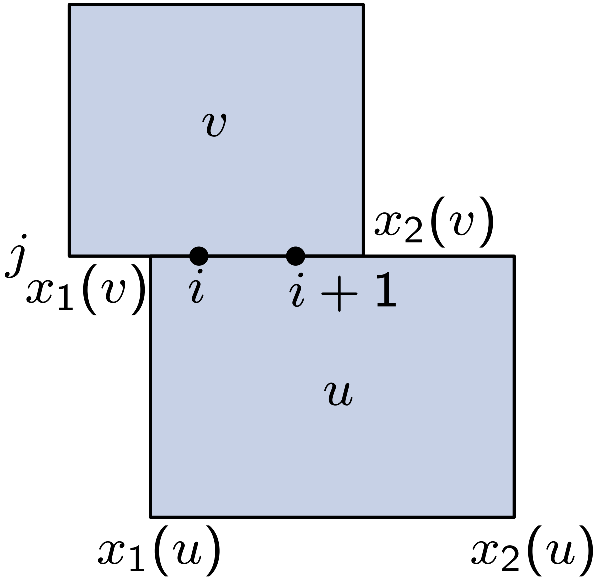
Lemma 8

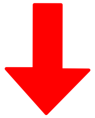
The assignment provided by the algorithm has the following property: rectangles assigned to vertices u and v have a common segment if and only if there exists edge (u, v) in the graph.

Proof:

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.





Rectangular Dual

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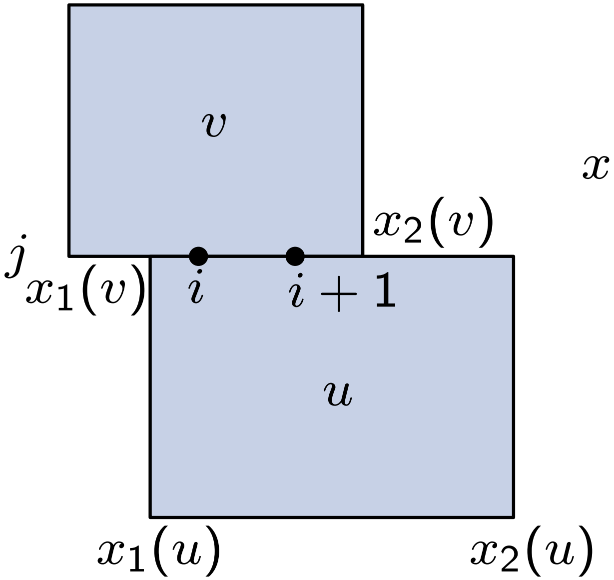
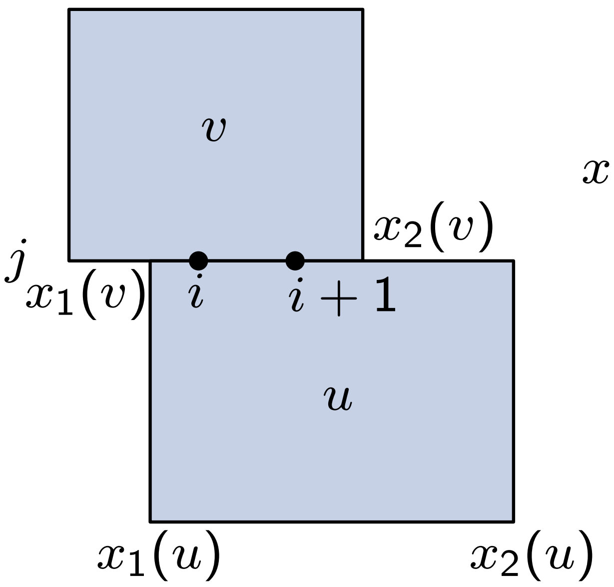


$x_1(v) \leq i,\, i+1 \leq x_2(v)$ and $x_1(u) \leq i,\, i+1 \leq x_2(u)$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$\Downarrow$

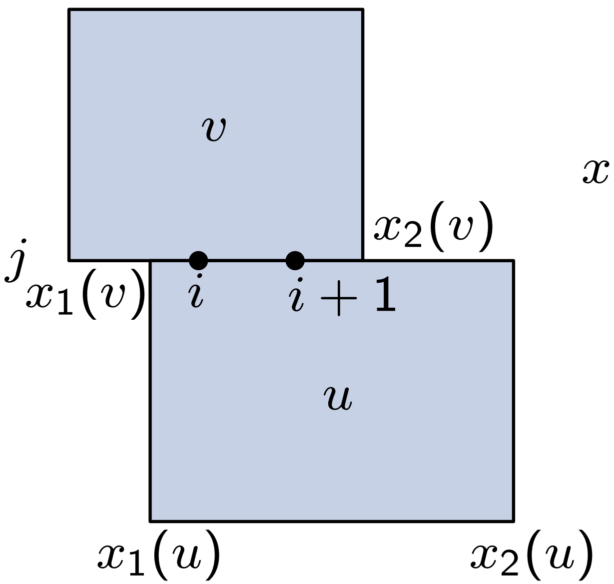
$x_1(v) \leq i,\ i+1 \leq x_2(v)$ and $x_1(u) \leq i,\ i+1 \leq x_2(u)$

$\Downarrow$ (Lemma 4)

u, v belong to P_i

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$x_1(v) \leq i,\ i+1 \leq x_2(v)$ and $x_1(u) \leq i,\ i+1 \leq x_2(u)$

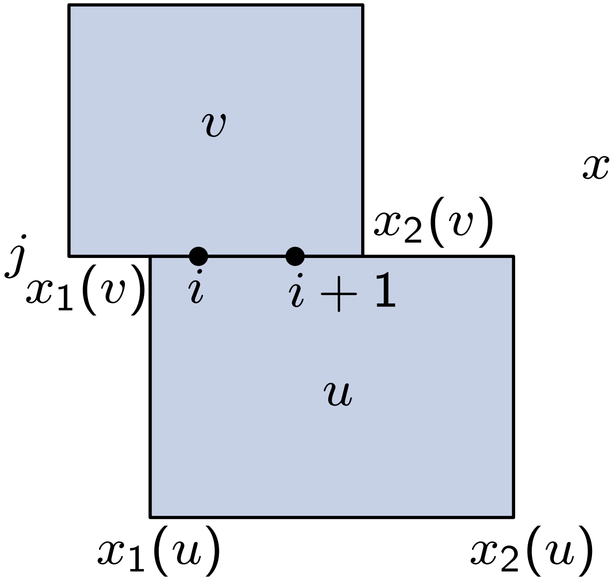
(Lemma 4)

u, v belong to P_i

If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$x_1(v) \leq i,\, i+1 \leq x_2(v)$ and $x_1(u) \leq i,\, i+1 \leq x_2(u)$

(Lemma 4)

u, v belong to P_i

If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

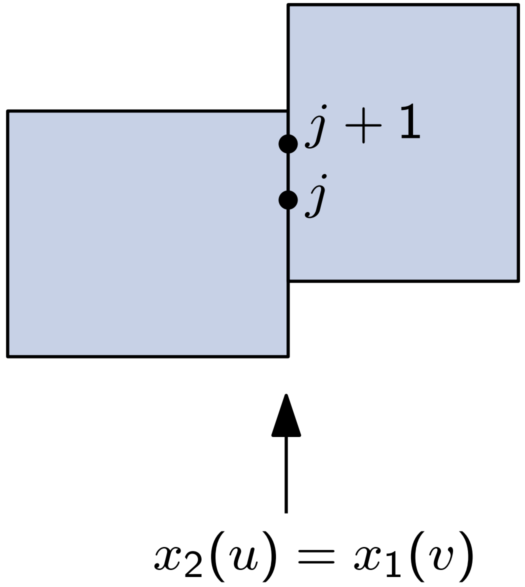
$$\Downarrow$$

- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$



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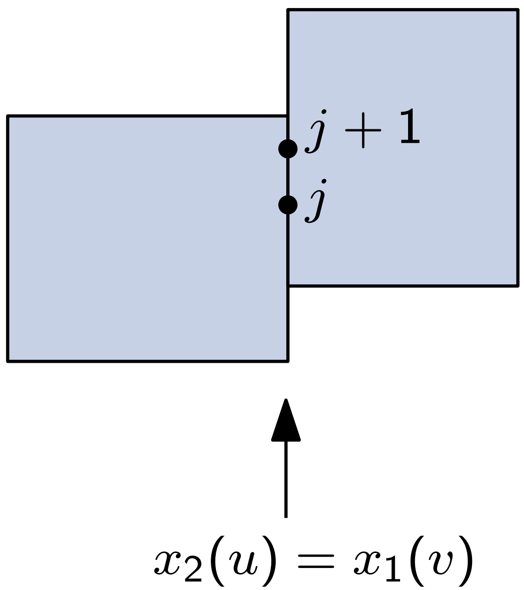

Lemma 8 is proved!

Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

Rectangular Dual

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Every PTP graph G has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of G
- Compute a revised canonical ordering of G
- Traverse the graph and color the edges, construct G_{S-N} and G_{E-W}
- Construct the duals $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$ of G_{S-N} and G_{E-W}, respectively
- Compute an st-numbering of $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$
- Assing coordinates to the rectangles representing vertices.