Exercises 1 - Convex Hulls & Line Segment Intersection

Discussion: Friday, May 4th, 2018

Lecture 1 – 18.04.2018

Exercise 1. Show that the last step of the algorithm $\text{FirstConvexHull}(P)$ can be implemented such that it has running time $O(n \log n)$.

\begin{align*}
\text{FirstConvexHull}(P) \\
E &\leftarrow \emptyset \\
\text{foreach } (p,q) \in P \times P \text{ with } p \neq q \text{ do} \\
&\quad \text{valid } \leftarrow \text{true} \\
&\quad \text{foreach } r \in P \text{ do} \\
&\quad\quad \text{if not } (r \text{ strictly right of } \overrightarrow{pq} \text{ or } r \in \overline{pq}) \text{ then} \\
&\quad\quad\quad \text{valid } \leftarrow \text{false} \\
&\quad\quad \text{if } \text{valid} \text{ then} \\
&\quad\quad\quad E \leftarrow E \cup \{(p,q)\}
\end{align*}

Construct sorted node list $L$ of $CH(P)$ from $E$

return $L$

Exercise 2. In the first lecture, we have seen the gift wrapping or Jarvis march algorithm for computing the convex hull of a set of $n$ points. Sketch a proof of the following theorem, in particular the correctness of the algorithm.

Theorem 1. The convex hull $CH(P)$ of a set of $n$ points $P$ in $\mathbb{R}^2$ can be computed in $O(n \cdot h)$ time using the gift wrapping algorithm, where $h = |CH(P)|$.

What degeneracies may occur in the input? How can you handle them correctly?
Figure 1: Convex polygon $P$ and the two tangents from an external point $p$.

Figure 2: Largest top-right region of $p$

**Exercise 3.** Let $P$ be a convex polygon with $n$ vertices and let $p$ be a point outside $P$ as shown in Figure 1.

Show that the two tangent lines at $P$ that pass through $p$ can be computed in $O(\log n)$ time, assuming that the polygon is given as a clockwise sorted list of its vertices. In which part of Chan’s convex hull algorithm is this subroutine needed?

**Exercise 4.** We require that any algorithm computing the convex hull of a given set of points returns the convex hull vertices as a clockwise sorted list of points.

1. Show that any algorithm for computing the convex hull of $n$ points has a worst case running time of $\Omega(n \log n)$ and thus Graham Scan is worst-case optimal.
2. Why is the running time of the gift wrapping algorithm not in contradiction to part (a)?

**Lecture 2 – 25.04.2018**

**Exercise 5.** The algorithm seen in the lecture for finding the intersection points of $n$ line segments required $O(n + I)$ space, where $I$ is the number of intersection points. Modify the algorithm to use $O(n)$ space only and argue that the modified algorithm remains correct. Does this affect the asymptotic running time?

**Exercise 6.** Let $P$ be a set of $n$ points in the plane. The *largest top-right region* of a point $p \in P$ is the union of all open axis-aligned squares that touch $p$ with their bottom left corner and contain no other point of $P$ in their interior.
(1) Prove that the largest top-right region of a point is either a square or the intersection of two open half-planes.

(2) Let \( p \) be a point in \( P \), let \( O_1 \) be the upper octant and \( O_2 \) be the right octant of \( p \); see Fig. 2. Which point in \( O_1 \cap P \) restricts the largest top-right region of \( p \) at most? Which point in \( O_2 \cap P \) restricts the largest top-right region of \( p \) at most?

(3) Describe an algorithm that computes for all points in \( P \) the largest top-right region using \( O(n \log n) \) running time in total.

\textit{Hint:} Determine the points of question (2) using two sweeps.